Pathspace Representation of Affine Processes

Nicoletta Gabrielli

ETH Zürich

nicoletta.gabrielli@math.ethz.ch

April 16, 2013

Option Pricing

The problem:

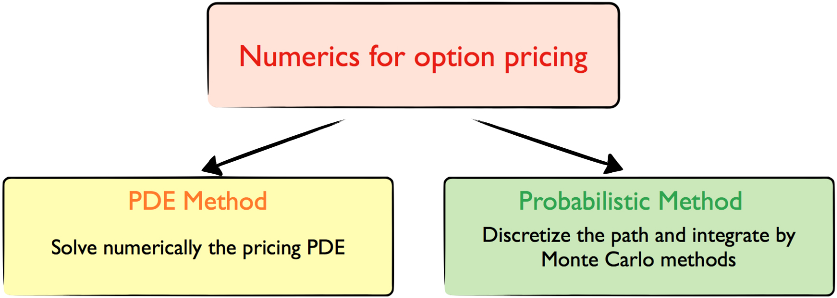
given

- $(X_t^x)_{t\in[0,T]}$ stock process
- H payoff function, possibly depending on the whole path up to time T

find $\mathbb{E}^x\Big[H(X_t,\ t\in[0,T])\Big]$.

Example

$$\begin{aligned} H:\mathcal{D}(\mathbb{R}_{\geq 0};D) &\to \mathbb{R}_{\geq 0} \\ X^x &\mapsto H(X^x) := \max\left(X_T^x - K, 0\right) \\ X^x &\mapsto H(X^x) := \max\left(\int_0^T X_t^x dt - K, 0\right) \end{aligned}$$


Numerics for option pricing
PDE Method
Solve numerically the pricing PDE
Probabilistic Method
Discretize the path and integrate by Monte Carlo methods

Approximation of trajectories by MC methods

step 1 Fix a uniform partition in $[0, T]$
$\{t_0 = 0, \dots, t_k = kh, \dots, t_N = T\}, \quad h = \frac{T}{N}$.

step 2 Find a piecewise constant approximating process $(\widehat{X}_{t_k})_{k=0,\dots,N}$ such that

- $\widehat{X}_{t_0} = x$,
- $\widehat{X}$ is a weak $\nu-$order approximation of X^x.

Definition

For every $f \in C^\infty$ with compact support there exists a $K > 0$ such that

$$\left|\mathbb{E}^x\left[f(X_T)\right] - \mathbb{E}^x\left[f(\widehat{X}_{t_N})\right]\right| \leq Kh^\nu.$$

Find the approximating sequence

Original process: X^x a time-homogeneous Markov process with state space D and associated semigroup

$$P_t f(x) := \mathbb{E}^x\Big[f(X_t)\Big], \quad f \in \mathcal{M},$$

$\mathcal{M}$ space of measurable functions where $\mathbb{E}^x\Big[f(X_t)\Big]$ is well defined.

Approximated process: $\widehat{X}^x$ with transition semigroup Q_h which is bounded and

$$\lim_{N\to\infty} (Q_h)^N f = P_T f.$$

If $\widehat{X}$ is a good approximation of X we need

1. $(Q_h)^k$ to be close to P_{t_k},
2. short time asymptotic

$$|Q_h P_s f - P_h P_s f| \leq K\rho(x) h^{\nu+1}, \text{ for all } f \in \mathcal{M}, \ h, s \in [0, T],$$

3. $P_t \mathcal{M} \subseteq \mathcal{M}$ to do the iteration.

From single step to multi-step

$$(Q_h)^N - P_T f = \sum_{k=1}^{N-1} (Q_h)^{N-k} (Q_h - P_h) P_{t_k} f.$$

Follow the lines of ODE approximation

If $f \in C^2$ with bounded derivative, from Dynkin formula

$$\mathbb{E}^x\Big[f(X_h)\Big] = f(x) + h\int_0^1 \mathbb{E}^x\Big[\mathcal{A}f(X_{sh})\Big]ds.$$

Iterating...

$$\begin{aligned}\mathbb{E}^x\Big[f(X_h)\Big] =& f(x) + \sum_{k=1}^{\nu}\frac{h^k}{k!}\mathcal{A}^k f(x) \\ &+ \frac{h^{\nu+1}}{\nu!}\int_0^1 (1-s)^{\nu}\,\mathbb{E}^x\Big[\mathcal{A}^{\nu+1}f(X_{sh})\Big]ds\end{aligned}$$

How far can we go with the space of test function $\mathcal{M}$?

The choice of test functions

Take $\mathcal{M} = C^{\infty}_{\mathrm{pol}}$ (see [Jacod et al., 2005], [Alfonsi, 2010])

Definition (The function space C^k_{pol})

A function $f \in C^k_{\mathrm{pol}}$ if

- $f \in C^k$
- for all α multi-index with $|\alpha| \leq k$, there exist constants C_α and e_α such that

$$|\partial^\alpha f(x)| \leq C_\alpha (1 + |x|^{e_\alpha}), \quad \text{for all } x \in D.$$

Theorem (Theorem 1.9 in [Alfonsi, 2010])

Under the assumptions

1. *(Uniform bounded moments)*
2. *($\nu-$ order asymptotic of the local error)*
3. *(Regularity Kolmogorov PIDE) f is a function such that $u(t,x) = \mathbb{E}^x\left[f(X_{T-t})\right]$ is the solution of the PIDE*

$$\partial_t u(t,x) + \mathcal{A}u(t,x) = 0 \ \ \textit{with} \ \ u(t,x) \in C^k_{pol}$$

Then $\widehat{X}^x$ is a weak $\nu-$order scheme for X^x.

Two types of error

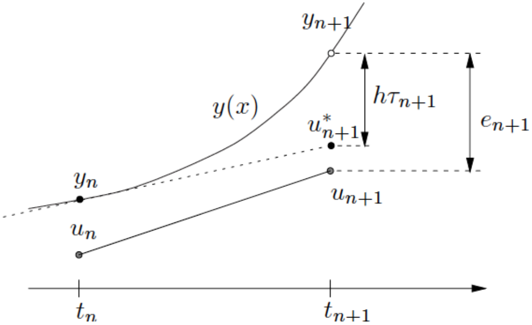


Figure : Geometrical interpretation of the local truncation error and the true error. The picture in taken from [Quarteroni et al., 2010]

Problem 1

Find Q_h such that

$$\lim_{N\to\infty} (Q_{\frac{t}{N}})^N f = P_t f.$$

High dimensional state space

$\hookrightarrow$ High order numerical schemes

Domain constrains

$\hookrightarrow$ Geometry preserving schemes

Problem 2

Is the stability condition satisfied?

Feynman-Kac representation

If u is a classical solution of

$$\partial_t u(t,x) + \mathcal{A}u(t,x) = 0$$

and its derivatives are bounded by a polynomial function uniformly in t, then it has the probabilistic representation

$$u(t,x) = \mathbb{E}^x\Big[f(X_{T-t})\Big].$$

$\hookrightarrow$ Is u smooth enough?

Regularity of the solution of the Kolmogorov PIDE

- Regularity of the Kolmogorov equation in time is easier to handle(Dynkin formula) than regularity in space.
- Idea: Switch time and space, what do you get?

Example: One dimensional case in $\mathbb{R}_{\geq 0}$

$$(X_t^x)_{t\geq 0}$$

Regularity of the solution of the Kolmogorov PIDE

- Regularity of the Kolmogorov equation in time is easier to handle(Dynkin formula) than regularity in space.
- Idea: Switch time and space , what do you get?

Example: One dimensional case in $\mathbb{R}_{\geq 0}$

$$({X^{x}}_{t})_{x \geq 0}$$

Pathspace representation

$$X_t^x := x + 2\int_0^t \sqrt{X_s}\,dW_s$$

$$L_x^t := 2t \sum_{k=1}^{N_{x/(2t)}} \mathbb{e}_k$$

$$X_t^x \overset{d}{=} L_x^t$$

Pathspace representation

$$X_t^x := x + 2\int_0^t \sqrt{X_s}\, dW_s$$

$$L_x^t := 2t \sum_{k=1}^{N_{x/(2t)}} \mathbb{e}_k$$

$$X_t^x \stackrel{d}{=} L^t{}_x$$

Pathspace representation

$$X^{x}{}_{t} := x + 2\int_0^t \sqrt{X_s}\, dW_s$$

$$L_x^t := 2t \sum_{k=1}^{N_{x/(2t)}} \mathbb{e}_k$$

$$X^{x}{}_{t} \overset{d}{=} L_x^t$$

References

Bouleau, N. and Chateau, O.
Le processus de la subordination.
(1989)

Bouleau, N.
Stochastic approach for the subordination in Bochner sense.
(2009)

Pitman, J. and Yor, M.
A decomposition of Bessel Bridges.
(1982)

Structure of the talk

Affine processes

From Affine Processes to Lévy Processes and back

Path approximation by Time-Space transformations

Regularity solution Kolmogorov PIDE

Conclusions

Outline

Affine processes

From Affine Processes to Lévy Processes and back

Path approximation by Time-Space transformations

Regularity solution Kolmogorov PIDE

Conclusions

Affine processes on the canonical state space

An *affine processes* on the canonical state space $D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ is a time homogeneous Markov process

$$X = (\Omega, (\mathcal{F}_t)_{t\geq 0}, (p_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{P}^x)_{x\in D_\Delta})$$

satisfying the following properties:

- (*stochastic continuity*) for every $t \geq 0$ and $x \in D$, $\lim_{s\to t} p_s(x,\cdot) = p_t(x,\cdot)$ weakly,
- (*affine property*) there exist functions $\boldsymbol{\phi} : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}$ and $\Psi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}^d$ such that

$$\mathbb{E}^x\left[e^{\langle u, X_t\rangle}\right] = \int_D e^{\langle u,\xi\rangle} p_t(x, d\xi) = e^{\boldsymbol{\phi}(t,u) + \langle \Psi(t,u), x\rangle}$$

for all $x \in D$ and $(t,u) \in \mathbb{R}_{\geq 0} \times \mathcal{U}$, with

$$\mathcal{U} = \left\{ u \in \mathbb{C}^d \mid e^{\langle u,x\rangle} \text{ is a bounded function on } D \right\}. \quad (1)$$

From affine processes to linear processes

Let $AP(D)$ be the space of affine processes with state space D. Define the map

$$\begin{aligned} \infty : AP(\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n) &\rightarrow AP(\mathbb{R}^{m+1}_{\geq 0} \times \mathbb{R}^n) \\ X &\mapsto X^\infty \end{aligned}$$

Input: X with $\mathbb{E}^x\left[e^{\langle u, X_t\rangle}\right] = e^{\phi(t,u)+\langle x, \Psi(t,u)\rangle}$

Output: X^∞ with $\mathbb{E}^{(1,x)}\left[e^{\left\langle u, X_t^\infty\right\rangle}\right] = e^{\langle (1,x), \Psi^\infty(t,u)\rangle}$ where

$$\Psi^\infty(t, u_0, u_1, \ldots, u_d) := \begin{pmatrix} \phi(t, u_1, \ldots, u_d) + u_0 \\ \Psi(t, u_1, \ldots, u_d) \end{pmatrix}$$

Linear structure

There exists a function $\Psi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}^d$ such that

$$\mathbb{E}^x\left[e^{\langle u, X_t\rangle}\right] = \int_D e^{\langle u,\xi\rangle} p_t(x, d\xi) = e^{\langle \Psi(t,u), x\rangle}$$

Generalized Riccati equations

On the set $\mathcal{Q} = \mathbb{R}_{\geq 0} \times \mathcal{U}$, the function Ψ satisfies the following system of generalized Riccati equations:

Generalized Riccati equations

$$\partial_t \Psi(t,u) \quad = \mathbf{R}(\Psi(t,u)), \quad \Psi(0,u) = u$$

where for each $k = 1, \ldots, d$ the function $\mathbf{R}_k$ has the following Lévy-Khintchine form

$$\begin{aligned} \mathbf{R}_k(u) \quad &= \quad \langle \beta_k, u \rangle + \frac{1}{2} \langle u, \alpha_k u \rangle - \gamma_k \\ &+ \int_{D \setminus \{0\}} \left(e^{\langle u, \xi \rangle} - 1 - \left\langle \pi_{J \cup \{k\}} u, \pi_{J \cup \{k\}} h(\xi) \right\rangle \right) M_k(d\xi). \end{aligned}$$

here

Infinite divisibility in D

Let $\mathcal{C}$ the *convex cone* of continuous function $\eta : \mathcal{U} \to \mathbb{C}_+$ of type

$$\eta(u) = \langle \mathrm{b}, u \rangle + \frac{1}{2} \langle \pi_J u, \sigma \pi_J u \rangle + \int_{D \setminus \{0\}} \left(e^{\langle u, \xi \rangle} - 1 - \langle \pi_J u, \pi_J h(\xi) \rangle \right) \nu(d\xi). \qquad (*)$$

Definition

A distribution λ on D_Δ is *infinitely divisible* if and only if its Laplace transform takes the form $e^{\eta(u) - \mathrm{c}}$, where η has the form $(*)$ and $\mathrm{c} = \log \lambda(D)$.

$p_t(x, \cdot)$ is infinitely divisible in D!

Outline

From affine processes to Lévy processes

Let $(\mathcal{D}(\mathbb{R}_{\geq 0}; D_\Delta), (\mathcal{F}_t)_{t\geq 0}, (p_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{P}^x)_{x\in D_\Delta})$ be a linear process on the canonical state space $D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$.

Proposition

For each fixed $t > 0$ and $x \in D \setminus \{0, \Delta\}$, there exists process $(L^t_{sx})_{s\in[0,1]}$ such that:

1. $L^t_0 = 0$,
2. for every $0 \leq s_1 \leq s_2 \leq 1$, the increment $L^t_{s_2 x} - L^t_{s_1 x}$ is independent of the family $(L^t_{sx})_{s\in[0,s_1]}$ and it distributed as $X_t^{(s_2 - s_1)x}$.

Moreover for any fixed $t \geq 0$, and $x \in D$ there exists a unique modification $\widetilde{L}^t$ of L^t which is a Lévy process with càdlàg paths.

The proof

Apply Kolmogorov's existence Theorem with the convolution semigroup $(p_t(sx, \cdot))_{s \geq 0}$.

Chapman-Kolmogorov's equations → Semigroup property in time

$p_{s+t}(x, \cdot) = p_s \cdot p_t(x, \cdot) := \int p_t(x, dy) p_s(y, \cdot)$,

Linearity → Convolution property in space

$p_t(x + y, \cdot) = p_t(x, \cdot) * p_t(y, \cdot)$.

Let t run

Remark: Here t appears here as a parameter of $(L^t_{sx})_{s\in[0,1]}$.

Idea: Let t evolve and consider the above construction on the path space.

Result: Construct a path valued process $(L^{\cdot}_{sx})_{s\in[0,1]}$ which starts in zero and it reaches $(X^x_t)_{t\geq 0}$ at time 1. here

Make things rigorous

Theorem

There exists a process $(L_{sx})_{s\geq 0}$ *taking values in* $\mathcal{D}(\mathbb{R}_{\geq 0}; D_\Delta)$ *such that*

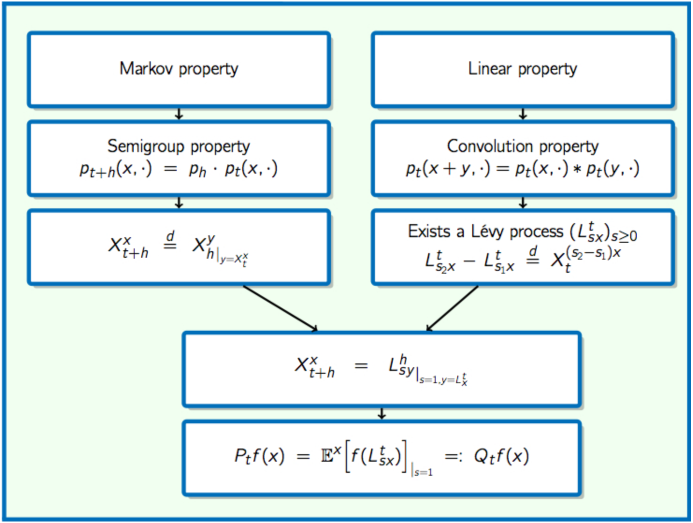
1. *it has stationary and independent increments,*
2. *it is stochastically continuous,*
3. *it holds*

$$\mathbb{E}^x\left[e^{\langle u, X_t\rangle}\right] = e^{\langle x, \Psi(t,u)\rangle} = \mathbb{E}\left[e^{\langle u, \mathrm{ev}_t(L_{sx})\rangle}\right]\Big|_{s=1}$$

Proof: Apply Kolmogorov's existence Theorem with the convolution semigroup $\wp_s(x,\cdot) := \mathbb{P}^{sx},\ s \geq 0$.

Outline

A guiding principle



Lévy-Kintchine decomposition of an affine process

step 1 Solve $\Psi(t,u) = u + \int_0^t \mathbf{R}(\Psi(s,u))ds, \quad u \in \mathcal{U}.$

step 2 Do Lévy-Khintchine decomposition of $\Psi(t,u)$.

Approximation Riccati equations

1. Fix $N > 0$, $T > 0$ and a partition $\{t_0 = 0, t_1 = h, \ldots, t_N = T\}$ with $h = \frac{T}{N}$.
2. Let y_n be the approximation of $\Psi(t_n, u)$.

Example: Forward Euler's method

For small h,

$$\mathbb{E}^x\left[e^{\langle u, X_h\rangle}\right] = e^{\langle x,u\rangle + h\langle x, \mathrm{R}(u)\rangle}, \quad u \in \mathcal{U}.$$

- ! X_t^x is infinitely divisible in $D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ but $\langle x, \mathbf{R}(u)\rangle$ has the Lévy-Khintchine representation in $\mathbb{R}^d$! here

⇝ Seek methods that preserve positivity of the semiflow.

Example 1: Feller diffusion ⇜⇝ homographic ray

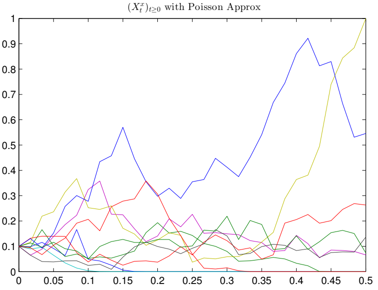
$$\mathbb{E}\left[e^{u\mathcal{L}_x^t}\right] = \exp\left(x\frac{u}{1-\frac{ut}{2}}\right), \quad \mathcal{R}e(u) < \frac{2}{t},$$

Affine process on $\mathbb{R}_{\geq 0}$ with functional characteristic $\mathbf{R}(u) = \frac{1}{2}u^2$.

Exact approximation

$$\begin{aligned} \widehat{X}_0^x &= x, \\ \widehat{X}_{t_{k+1}}^x &= (\mathcal{L}_{sx}^h)_{s=1, x=\widehat{X}_{t_k}^x}, \ k \geq 0 \end{aligned}$$

where $\mathcal{L}_{\cdot x}^t$ is a subordinator with $\nu(t,x,d\xi) := \frac{4x}{t^2}e^{-\frac{2\xi}{t}}d\xi$.



Example 2: Neveu Branching $\leftrightsquigarrow$ stable ray

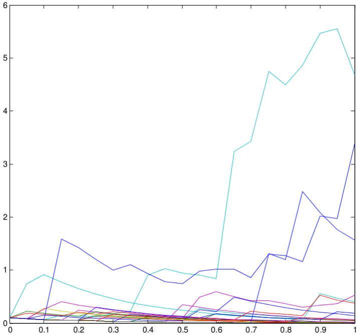
$$\mathbb{E}\left[e^{u\mathcal{J}_x^t}\right] = \exp\left(-x(-u)^{e^{-t}}\right)$$

Affine process on $\mathbb{R}_{\geq 0}$ with functional characteristic $\mathbf{R}(u) = -u\log(-u)$.

Exact approximation

$$\begin{aligned} \widehat{X}_0^x &= x \\ \widehat{X}_{t_{k+1}}^x &= (\mathcal{J}_{sx}^h)_{s=1, x=\widehat{X}_{t_k}^x}, \ k \geq 0. \end{aligned}$$

where $\mathcal{J}_{\cdot x}^t$ is a subordinator with $\nu(t, x, d\xi) := \frac{xe^{-t}}{\Gamma(1-e^{-t})}\xi^{-1-e^{-t}}d\xi$.



CIR with jumps

The Model:

$$dZ_t^z = cZ_t^z dt + \sqrt{Z_t^z}\, dW_t + \int_{|\xi|>1} \xi N(ds, d\xi)$$
$$+ \int_{|\xi|\le 1} \xi \left(N(dt, d\xi) - \frac{d\xi}{\xi^2} Z_t^z dt\right)$$

$$R^{C+J}(u) = \boxed{cu + \tfrac{1}{2}u^2} + \boxed{\int_0^\infty \left(e^{u\xi} - 1 - u\xi \mathbb{1}_{|\xi|\le 1}\right) \tfrac{d\xi}{\xi^2}}$$

Idea: Split the vector field

$$R^{C+J}(u) = \underbrace{R^C(u)}_{\text{Feller Diffusion}} + \underbrace{R^J(u)}_{\text{Neveu CSBP}}$$

Geometry preserving schemes

1. Let $\widehat{\Psi}(h,u) := \Psi^J(h, \Psi^C(h,u))$.
2. Define

$$\begin{aligned} y_0(u) &= u \\ y_{n+1}(u) &= \widehat{\Psi}(h, y_n(u)) \qquad n = 0, \ldots, N-1. \end{aligned}$$

3. $\lim_{n\to\infty} y_N(u) =: \Psi^{C+J}(t,u)$ with

$$\Psi^{C+J}(t,u) = u + \int_0^t R^{C+J}(\Psi^{C+J}(s,u))ds.$$

Remark: $P_T f = \lim_{N\to\infty}(Q_h f)^N, \qquad Q_h := P_h^C P_h^J$. Other examples: main.

Outline

Set the notation

Consider

$$u(t,x) = \mathbb{E}^x\Big[f(X_{T-t})\Big] \overset{\text{not.!}}{=} \mathbb{E}\Big[f(L_s^{(T-t,x)})\Big]\Big|_{s=1}.$$

Introduce

$$v^{(t,x)}(s,y) = \underbrace{\mathbb{E}^y\Big[f(L_{(1-s)}^{(T-t,x)})\Big]}_{\text{it evolves in time } s} = \mathbb{E}\Big[f(L_{(1-s)}^{(T-t,x)} + y)\Big].$$

Remark: Regularity in x for $u(t,x)$ can be obtained by regularity in s of $v^{(t,x)}(s,y)$.

Change the perspective

General case: $(L_s^{(t,x)})_{s\geq 0}$ is a Lévy process with infinitesimal generator

$$\begin{aligned}\mathcal{L}^{(t,x)}f(y) &= \frac{1}{2}\operatorname{Tr}(\sigma(t,x)\triangle f(y)) + \langle \mathrm{b}(t,x), \nabla f(y)\rangle \\ &+ \int (f(y+\xi) - f(y) - \langle \xi, \nabla f(y)\rangle)\, \nu(t,x,d\xi).\end{aligned}$$

Processes on $\mathbb{R}^m_{\geq 0}$: $(L_s^{(t,x)})_{s\geq 0}$ is a subordinator.

Feller diffusion: $(L_s^{(t,x)})_{s\geq 0}$ is a compound Poisson process with Lévy measure

$$\nu(t,x,d\xi) = \frac{x}{(2t)^2}\exp\left(-\frac{\xi}{2t}\right)$$

Regularity of the solution of the Kolmogorov PIDE

Advantages

- If $\mathcal{L}^{(t,x)} C^{\infty}_{\mathrm{pol}} \subseteq C^{\infty}_{\mathrm{pol}}$ all the derivatives with respect of s exists.
- In the diffusive case regularity of $(s, y) \mapsto v^{(t,x)}(s, y)$ has been proved already in [Talay and Tubaro, 1990].
- The Lévy case has been studied in [Jacod et al., 2005] and [Protter and Talay, 1997].

A one dimensional example: The Feller diffusion

$$X^{x}{}_{t} := x + 2\int_0^t \sqrt{X_s}\, dW_s$$

$$L_x^t := 2t \sum_{k=1}^{N_{x/(2t)}} \mathbb{e}_k$$

$$X^{x}{}_{t} \overset{d}{=} L_x^t$$

- In [Alfonsi, 2005] regularity is obtained by using explicit expression of the density

Regularity for Kolmgorov's equation driven by operators of Lévy type

Idea Use results in [Jacod et al., 2005] for Lévy driven SDE.

$$v^{(t,x)}(s,y) := \mathbb{E}^y\left[f(L^{(T-t,x)}_{(1-s)})\right]$$

Sufficient conditions (Corollary 5.2. in [Protter and Talay, 1997])

- $f \in C^k_{\mathrm{pol}}$
- $\int_{|\xi|\geq 1} |\xi|^k \nu(t,x,d\xi) < \infty$

Then there exists a constant(depending on t and f) such that for all α multi-index with $|\alpha| \leq k$ there exists $M \in \mathbb{N}$ such that

$$|\partial^\alpha_{(s,y)} v^{(t,x)}(s,y)| \leq K(1+|y|^M).$$

Back to the Feller diffusion: $dX_t^x = 2\sqrt{X_t}dW_t$

Theorem (see Proposition 4.1. in [Alfonsi, 2005])

Given $f \in C_{pol}^k$, $\partial_x^\alpha u(t,x)$ is well defined and in C_{pol}^k for all $\alpha \in \mathbb{N}$. Moreover it is a classical solution of the PDE

$$\begin{aligned} \partial_t u(t,x) + 2x\partial_x^2 u(t,x) = 0, \\ u(T,x) = f(x). \end{aligned}$$

Proof:

For all $m \geq 0$ $\mathbb{E}^x\Big[(X_t)^m\Big] = \mathbb{E}\Big[(L_x^t)^m\Big] < \infty$.

$\hookrightarrow$ for all $l \geq 0$ $\partial_s^l v(s,y)$ is smooth and of polynomial growth

$\hookrightarrow$ for all $l \geq 0$ $\partial_x^k u(x,t)$ is smooth and of polynomial growth

Outline

Conclusions and future research

1. The understanding of linear processes as path valued Lévy processes leads to numerical schemes based on the approximation of **'easy to simulate'** processes.
2. **Higher order schemes** can be derived by performing a Strang splitting instead of the Lie-Trotter splitting.
3. This perspective allows us to consider **path dependent options** written on a stock driven by an Affine process as a European style option written on a path valued process.

Alfonsi, A. (2005).
On the discretization schemes for the CIR (and Bessel squared) processes.
Monte Carlo Methods Appl., 11(4):355–384.

Alfonsi, A. (2010).
High order discretization schemes for the CIR process: application to affine term structure and Heston models.
Math. Comp., 79(269):209–237.

Jacod, J., Kurtz, T. G., Méléard, S., and Protter, P. (2005).
The approximate Euler method for Lévy driven stochastic differential equations.
Ann. Inst. H. Poincaré Probab. Statist., 41(3):523–558.

Protter, P. and Talay, D. (1997).
The Euler scheme for Lévy driven stochastic differential equations.
Ann. Probab., 25:393–423.

Quarteroni, A., Sacco, R., and Saleri, F. (2010).
Numerical Mathematics.
Texts in Applied Mathematics. Springer.

Talay, D. and Tubaro, L. (1990).
Expansion of the global error for numerical schemes solving stochastic differential equations.
Stochastic Anal. Appl., 8(4):483–509 (1991).

Thank you

Lévy Khintchine decomposition of $\langle x, \Psi(t,u)\rangle$

For $i = 1, \dots, m$

$$\Psi_i(t,u) = \langle \mathsf{b_i}(\mathsf{t}), u\rangle + \frac{1}{2}\langle \pi_J u, \sigma_i(t)\pi_J u\rangle - \mathsf{c}_i(t) + \int_{D\setminus\{0\}} \left(e^{\langle u,\xi\rangle} - 1 - \langle \pi_J u, \pi_J h(\xi)\rangle\right)\nu_i(t, d\xi)$$

For $j = m+1, \dots, d$

$$\Psi_j(t,u) = \left\langle \mathsf{b_j}(\mathsf{t}), u\right\rangle .$$

Lévy Khintchine decomposition of $\langle x, \mathbf{R}(u)\rangle$

For $i = 1, \ldots, m$

$$\mathbf{R}_i(u) = \langle \beta_i, u\rangle + \frac{1}{2}\left\langle \pi_J u, \alpha^i_J \pi_J u\right\rangle + \frac{1}{2}\alpha^2_{i,i} u_i^2 - \gamma_i + \int_{D\setminus\{0\}} \left(e^{\langle u,\xi\rangle} - 1 - \left\langle \pi_{J\cup\{i\}} u, \pi_{J\cup\{i\}} h(\xi)\right\rangle\right) \nu_i(t, d\xi)$$

For $j = m+1, \ldots, d$

$$\mathbf{R}_j(u) = \left\langle \beta_j, u\right\rangle .$$

Back to main.

Construction

step1: Define the family of Markov kernels $(\wp_s(x,\cdot))_{s\geq 0}$

$$\wp_s(x,A) : (D_\Delta, \mathcal{B}(D_\Delta)) \rightarrow (\mathcal{D}(\mathbb{R}_{\geq 0}; D_\Delta), \mathcal{F}),$$

by $\wp_s(x,A) := \mathbb{P}^{sx}(A)$, $A \in \mathcal{F}$

- it is a convolution semigroup: $\wp_{s+t}(x,\cdot) = \mathbb{P}^{(s+t)x} = \mathbb{P}^{sx} * \mathbb{P}^{tx} = \wp_s(x,\cdot) * \wp_t(x,\cdot)$,
- $\wp_1 = \mathbb{P}^x$.

step2: $\mathcal{D}(\mathbb{R}_{\geq 0}; D_\Delta)^{[0,1]} := \{\varpi : [0,1] \rightarrow \mathcal{D}(\mathbb{R}_{\geq 0}; D_\Delta)\}$.

step3: for any *fixed* $x \in D$, define the map

$$\begin{aligned} L_{sx} : \mathcal{D}(\mathbb{R}_{\geq 0}; D_\Delta)^{[0,1]} &\rightarrow \mathcal{D}(\mathbb{R}_{\geq 0}; D_\Delta) \\ \varpi &\mapsto \varpi_s. \end{aligned}$$

step4: $(\mathcal{G}_s)_{s\in[0,1]}$ the natural filtration generated by L.

Kolmogorov's existence Theorem

There exists a probability measure P on $(\mathcal{D}(\mathbb{R}_{\geq 0}; D_\Delta)^{[0,1]}, \bigvee_{s\in[0,1]} \mathcal{G}_s)$ such that, for fixed $x \in D_\Delta$

$$P(L_{s_1 x} \in d\omega_1, \ldots, L_{s_n x} \in d\omega_n) = \bigotimes_{k=1}^{n} \wp_{s_k - s_{k-1}}(x, d\omega_k - \omega_{k-1}),$$

with $0 = s_0 \leq s_1 \leq \ldots \leq s_n \leq 1$.

- $(L_{sx})_{s\in[0,1]}$ a stochastic process taking values in $\mathcal{D}(\mathbb{R}_{\geq 0}; D_\Delta)$ with stationary and independent increments,
- In distribution L_{1x} coincide with X^x.
- It is stochastically continuous, i.e. $\lim_{s\to t} \wp_s(x, \cdot) = \wp_t(x, \cdot)$ weakly on $\mathcal{D}(\mathbb{R}_{\geq 0}; D_\Delta)$ for every $t \geq 0$ and $x \in D$.

Back to main.

Admissible parameters

For an affine process on $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ we can specify a set of admissible parameters

$$(b, \beta, a, \alpha, c, \gamma, m, M).$$

with

- $b, \beta_i \in \mathbb{R}^d,\ i = 1, \ldots, d,$
- $a, \alpha_i,\ i = 1, \ldots, d$ semi-definite positive matrices,
- $c, \gamma_i \in \mathbb{R}_{\geq 0},\ i = 1, \ldots, d,$
- $m, M_i,\ i = 1, \ldots, d$ Lévy measures,

where $d = n + m$.

Admissible parameters

This set of parameters is called *admissible* for $D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ if

diffusion	
$a_{k,h} = 0$	for $k \in I$ or $h \in J$,
$\alpha_j = 0$	for all $j \in J$,
$(\alpha_i)_{k,h} = 0$	if $k \in I \setminus \{i\}$ or $h \in I \setminus \{i\}$,

Admissible parameters

This set of parameters is called *admissible* for $D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ if

drift
$b \in D$,
$(\beta_i)_k \geq 0 \quad$ for all $i \in I$ and $k \in I \setminus \{i\}$,
$(\beta_j)_k = 0 \quad$ for all $j \in J$, $k \in I$

Admissible parameters

This set of parameters is called *admissible* for $D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ if

killing	
$\gamma_j = 0$	for all $j \in J$,

jumps	
$\operatorname{supp} m \subseteq D$	and $\int_{D\setminus\{0\}} \left((\lvert x_I\rvert + \lvert x_J\rvert^2) \wedge 1\right) m(dx) < \infty$,
$M_j = 0$	for all $j \in J$,
$\operatorname{supp} M_i \subseteq D$	for all $i \in I$ and
	$\int_{d\setminus\{0\}} \left((\lvert \pi_{I\setminus\{i\}}(x)\rvert + \lvert \pi_{J\cup\{i\}}(x)\rvert^2) \wedge 1\right) M_i(dx) < \infty$
	for all $i \in I$

Back to main.

Heston Model

$$dX_t^x = \sqrt{X_t^x}dB_t, \quad dY_t^y = -\frac{X_t^x}{2}dt + \sqrt{X_t^x}dW_t$$

$$R_1^H(u_1,u_2) = \frac{1}{2}\left(u_1^2 + u_2^2 - u_2\right), \quad R_2^H(u_1,u_2) = 0$$

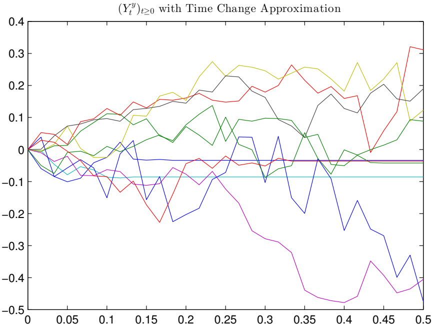
Idea: Solve the homogenous problem and then apply **nonlinear variation of constants.**

$$\begin{aligned}
\Psi_1^H(t,u_1,u_2) &= \underbrace{u_1 + \int_0^t R^C(\Psi_1^H(s,u_1,u_2))ds}_{\text{homogeneus part}} + tR_1^H(0,u_2) \\
&= \Psi^C(t,u_1) \\
&\quad + R_1^H(0,u_2)\underbrace{\int_0^t \partial_v \Psi^C(t-s,\Psi(s,u_1,u_2))ds}_{\text{Approximate by quadrature}}
\end{aligned}$$

The connection

Higher order approximation of the integral term coincides with higher approximation of the process.

Due to the representation of the Heston model as a time changed Brownian motion, one has $\mathbb{E}^{(x,0)}\left[e^{\langle u,Y_t\rangle}\right] = e^{x\Psi_I(t,R_1^H(0,u))}$, where $x\Psi_I$ is the logarithm of the Fourier–Laplace transform of the process I. The picture shows some approximated paths of Y when the path integral process I is approximated by composite trapezoid rule.



Back to main.