Affine processes from the perspective of path-space valued Lévy processes

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Structure of the talk

1. What is an affine process?
   - Examples
   - Definitions
2. Elementary transformations of AP
   - From affine processes to linear processes
   - From affine processes to semi–homogeneous affine processes
3. Pathwise construction of affine processes
   - Time–change techniques for affine processes
   - Applications

Outline

Examples of affine processes 1

Lévy processes

$$Y_t = \mathbf{y} + \boldsymbol{\mu}\mathbf{t} + \boldsymbol{\sigma}\mathbf{B}_{\mathrm{t}} + \int_0^t \int \xi 1_{\{|\xi| \leq 1\}}(J^Y(d\xi, ds) - m(d\xi)ds)$$
$$+ \int_0^t \int \xi 1_{\{|\xi| > 1\}} J^Y(d\xi, ds)$$

where (μ, α, m) is a Lévy triplet in $\mathbb{R}^n$, with $\alpha = \sigma\sigma^\top$.

Examples of affine processes 1

Lévy processes

$$Y_t = \mathbf{y} + \boldsymbol{\mu}\mathbf{t} + \boldsymbol{\sigma}\mathbf{B}_t + \int_0^t \int \xi 1_{\{|\xi|\leq 1\}} (J^Y(d\xi, ds) - m(d\xi)ds) + \int_0^t \int \xi 1_{\{|\xi|>1\}} J^Y(d\xi, ds)$$

where (μ, α, m) is a Lévy triplet in $\mathbb{R}^n$, with $\alpha = \sigma\sigma^\top$.

$$\mathbb{E}^y\left[e^{\langle u, Y_t\rangle}\right] = e^{t\eta(u)+\langle y, u\rangle}, \qquad u \in \mathrm{i}\mathbb{R}^n\,.$$

Examples of affine processes 2

Heston model

$$\begin{cases} V_t &= v + bt + \beta \int_0^t V_s ds + \varsigma \int_0^t \sqrt{V_s} dB_s^1 \\ Y_t &= y - \frac{1}{2} \int_0^t V_s ds + \int_0^t \sqrt{V_s} dB_s^2, \end{cases}$$

where

- $\beta, \varsigma \in \mathbb{R}$, $b \in \mathbb{R}_{\geq 0}$,
- $B = (B^1, B^2)$ is a Brownian motion in $\mathbb{R}^2$ with correlation .

Examples of affine processes 2

Heston model

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where

- $\beta, \varsigma \in \mathbb{R}$, $b \in \mathbb{R}_{\geq 0}$,
- $B = (B^1, B^2)$ is a Brownian motion in $\mathbb{R}^2$ with correlation .

$$\mathbb{E}^{(v,y)} \left[e^{u_1 V_t + u_2 Y_t} \right] = e^{\phi(t,u_1,u_2) + v\psi(t,u_1,u_2) + y u_2}, \quad (u_1, u_2) \in \mathrm{i}\mathbb{R}^2 .$$

Examples of affine processes 3

Bates model

$$\begin{cases} V_t &= v + bt + \beta \int_0^t V_s ds + \varsigma \int_0^t \sqrt{V_s} dB_s^1 \\ Y_t &= y - \frac{1}{2} \int_0^t V_s ds + \int_0^t \sqrt{V_s} dB_s^2 + \mathbf{J}_\mathrm{t}, \end{cases}$$

where

- J is a compound Poisson process

Examples of affine processes 3

Bates model

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where

- J is a compound Poisson process

$$\mathbb{E}^{(v,y)}\left[e^{u_1 V_t + u_2 Y_t}\right] = e^{\phi(t,u_1,u_2) + v\psi(t,u_1,u_2) + y u_2}, \quad (u_1, u_2) \in \mathrm{i}\mathbb{R}^2 .$$

In the above examples:

V stochastic variance process in $\mathbb{R}^m_{\geq 0}$,

Y (discounted) log–price process in $\mathbb{R}^n$,

and

- $X := (V, Y)$ is a time homogeneous Markov process in $D := \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$,
- there exist functions $\phi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}$ and $\Psi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}^d$ such that

$$\mathbb{E}^{(v,y)}\left[e^{\langle u_1, V_t\rangle + \langle u_2, Y_t\rangle}\right] = e^{\phi(t,u_1,u_2) + \langle v, \Psi(t,u_1,u_2)\rangle + \langle y, u_2\rangle},$$

for $u = (u_1, u_2) \in \mathcal{U}$, where $\mathcal{U} = \mathrm{i}\mathbb{R}^{m+n}$.

Outline

Definition

Let

$$(\Omega, (X_t)_{t\geq 0}, (\mathcal{F}_t^\natural)_{t\geq 0}, (p_t)_{t\geq 0}, (\mathbb{P}^x)_{x\in D})$$

be a time homogeneous Markov process. The process X is said to be an affine process if it satisfies the following properties:

- for every $t \geq 0$ and $x \in D$, $\lim_{s\to t} p_s(x, \cdot) = p_t(x, \cdot)$ weakly,
- there exist functions $\phi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}$ and $\Psi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}^d$ such that

  $$\mathbb{E}^x\left[e^{\langle u, X_t\rangle}\right] = \int_D e^{\langle u,\xi\rangle} p_t(x, d\xi) = e^{\phi(t,u)+\langle x,\Psi(t,u)\rangle},$$

  for all $x \in D$ and $(t, u) \in \mathbb{R}_{\geq 0} \times \mathcal{U}$.

Additional notation

Henceforth

$$\begin{aligned} D &= \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \subseteq \mathbb{R}^d \,, \\ D_\Delta &= D \cup \{\Delta\} \,, \\ \mathcal{U} &= \mathbb{C}^m_{\leq 0} \times \mathrm{i}\mathbb{R}^n \,, \\ I &= \{1, \ldots, m\} \,, \\ J &= \{m+1, \ldots, d\} \,. \end{aligned}$$

Given a set $H \subseteq \{1, \ldots, d\}$,

$$\begin{aligned} \pi_H : \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n &\to \mathbb{R}^H_{\geq 0} \\ x &\mapsto \pi_H x := (x_i)_{i \in H} \,. \end{aligned}$$

(Stochastic continuity + Affine property)

$\downarrow$[Cuchiero and Teichmann, 2011]

Càdlàg paths

$\downarrow$[Keller Ressel et al., 2011]

Regularity

Generalized Riccati equations

On the set $\mathcal{Q} = \mathbb{R}_{\geq 0} \times \mathcal{U}$, the functions ϕ and Ψ satisfy the following system of generalized Riccati equations:

$$\begin{aligned} \partial_t \phi(t,u) &= F(\Psi(t,u)), \quad \phi(0,u) = 0, \\ \partial_t \Psi(t,u) &= R(\Psi(t,u)), \quad \Psi(0,u) = u\,. \end{aligned}$$

Lévy–Khintchine form for the vector fields

Theorem

The functions F and R_k, for each $k = 1, \dots, d$, have the following Lévy-Khintchine form

$$
\begin{aligned}
F(u) &= \langle b, u \rangle + \frac{1}{2} \langle u, au \rangle - c \\
&\quad + \int_{D\setminus\{0\}} \left(e^{\langle u,\xi \rangle} - 1 - \langle \pi_J u, \pi_J h(\xi) \rangle \right) m(d\xi), \\
R_k(u) &= \langle \beta_k, u \rangle + \frac{1}{2} \langle u, \alpha_k u \rangle - \gamma_k \\
&\quad + \int_{D\setminus\{0\}} \left(e^{\langle u,\xi \rangle} - 1 - \left\langle \pi_{J\cup\{k\}} u, \pi_{J\cup\{k\}} h(\xi) \right\rangle \right) M_k(d\xi).
\end{aligned}
$$

Admissible parameters

For an affine process on $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ we can specify a set of admissible parameters

$$(b, \beta, a, \alpha, c, \gamma, m, M)$$

with

- $b, \beta_i \in \mathbb{R}^d$, $i = 1, \ldots, d$,
- a, α_i, $i = 1, \ldots, d$ semi-definite positive matrices,
- $c, \gamma_i \in \mathbb{R}_{\geq 0}$, $i = 1, \ldots, d$,
- m, M_i, $i = 1, \ldots, d$ Lévy measures,

where $d = n + m$.

diffusion	
$a_{kl} = 0$	for $k \in I$ or $l \in I$,
$\alpha_j = 0$	for all $j \in J$,
$(\alpha_i)_{kl} = 0$	if $k \in I \setminus \{i\}$ or $l \in I \setminus \{i\}$,

drift	
$b \in D$,	
$(\beta_i)_k \geq 0$	for all $i \in I$ and $k \in I \setminus \{i\}$,
$(\beta_j)_k = 0$	for all $j \in J$, $k \in I$,

killing	
$\gamma_j = 0$	for all $j \in J$,

jumps	
$\operatorname{supp} m \subseteq D$	and $\int \big((\lvert\pi_I \xi\rvert + \lvert\pi_J \xi\rvert^2) \wedge 1 \big)\, m(d\xi) < \infty$,
$M_j = 0$	for all $j \in J$,
$\operatorname{supp} M_i \subseteq D$	for all $i \in I$ and
	$\int \big((\lvert\pi_{I \setminus \{i\}} \xi\rvert + \lvert\pi_{J \cup \{i\}} \xi\rvert^2) \wedge 1 \big) M_i(d\xi) < \infty$.

Some remarks

- The theory of affine processes has being dominated by **weak constructions**.
- Stochastic continuity and the affine property are sufficient for the existence of a **version with càdlàg trajectories**, which can be defined on the canonical probability space of càdlàg paths.

Here:

- We given an alternative construction of affine property in "strong sense".
- The existence result relies on a **time–space transformation** of Lévy trajectories.
- Càdlàg property follows directly from the construction.

Theorem (Theorem 3.4 in [Kallsen, 2006])

Let X be an affine process with set of admissible parameters $(b, \beta, a, \alpha, 0, 0, m, M)$. On a possibly enlarged probability space, there exist $d+1$ independent $\mathbb{R}^d$-valued Lévy processes $Z^{(k)}$, $k = 0, \ldots, d$ such that

$$X_t \overset{law}{=} x + Z_t^{(0)} + \sum_{k=1}^{d} Z^{(k)}_{\int_0^t X_r^{(k)} dr} .$$

Question: Is it possible to construct X in a pathwise sense?

Outline

Let $AP(D)$ be the space of affine processes with state space D.
Define the map

$$\begin{aligned} \text{ꝏ} : AP(\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n) &\rightarrow AP(\mathbb{R}^{m+1}_{\geq 0} \times \mathbb{R}^n) \\ X &\mapsto X^{\text{ꝏ}} \end{aligned}$$

Input: X with $\mathbb{E}^x\left[e^{\langle u, X_t \rangle}\right] = e^{\phi(t,u) + \langle x, \Psi(t,u) \rangle}$

Output: $X^{\text{ꝏ}}$ with $\mathbb{E}^{x^{\text{ꝏ}}}\left[e^{\left\langle u, X^{\text{ꝏ}}_t \right\rangle}\right] = e^{\langle x^{\text{ꝏ}}, \Psi^{\text{ꝏ}}(t,u) \rangle}$

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From AP to semi–homogeneous AP

Theorem 5.1 in [Keller-Ressel et al., 2011]

Let X be an affine process. Recall that, there exists $\mathcal{B}_J \in \mathbb{R}^{n\times n}$ such that $\pi_J R(u) = \mathcal{B}_J^\top u$. Define the matrix

$$T = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & \mathcal{B}_J^\top \end{array} \right) \in \mathbb{R}^{d\times d}$$

and the map

$$\begin{aligned} \mathcal{T} : AP(D) &\rightarrow AP(D) \\ X &\mapsto X - T^\top \int_0^\cdot X_s ds \,. \end{aligned}$$

The map $\mathcal{T}$ is a bijection between affine processes of with $\pi_J R(u) = \mathcal{B}_J^\top u$ and the class of affine processes with $\pi_J R(u) = 0$.

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The result

Theorem [G. and Teichmann, 2014b]

Let (F, R) be a couple of functional characteristics such that $F = 0$ and $\pi_J R = 0$. Let $Z^{(1)}, \ldots, Z^{(\mathrm{m})}$ be independent $\mathbb{R}^d$-valued Lévy processes with

$$\mathbb{E}\left[e^{\left\langle u, Z_t^{(k)} \right\rangle}\right] = e^{tR_k(u)}, \qquad u \in \mathcal{U}\,.$$

Then the time–change equation

$$X_t = x + \sum_{k=1}^{\mathrm{m}} Z^{(k)}_{\int_0^t X_r^{(k)} dr}$$

admits a unique solution, which is an affine process with respect to the **time–changed filtration**.

Multiparameter time–change filtration

- Define
  $Z = (Z_1^{(1)}, \dots, Z_d^{(1)}, \dots, Z_1^{(m)}, \dots, Z_d^{(m)}) =: (Z^{(1)}, \dots, Z^{(md)})$.
- For all $\underline{s} = (s_1, \dots, s_{md}) \in \mathbb{R}_{\geq 0}^{md}$
  $\mathcal{G}_{\underline{s}}^{\natural} := \sigma\left(\{Z_{t_h}^{(h)},\ t_h \leq s_h,\ \text{for } h = 1, \dots, md\}\right)$.
- Complete it by $\mathcal{G}_{\underline{s}} = \bigcap_{n\in\mathbb{N}} \mathcal{G}^{\natural}_{\underline{s}^{(n)}+\frac{1}{n}} \vee \sigma(\mathcal{N})$.

Definition

A random variable $\underline{\tau} = (\tau_1, \dots, \tau_{md}) \in \mathbb{R}_{\geq 0}^{md}$ is a $(\mathcal{G}_{\underline{s}})$-stopping time if

$$\{\underline{\tau} \leq \underline{s}\} := \{\tau_1 \leq s_1, \dots, \tau_{md} \leq s_{md}\} \in \mathcal{G}_{\underline{s}},\ \text{for all } \underline{s} \in \mathbb{R}_{\geq 0}^{md}.$$

If $\underline{\tau}$ is a stopping time,

$$\mathcal{G}_{\underline{\tau}} := \{B \in \mathcal{G} \mid B \cap \{\underline{\tau} \leq \underline{s}\} \in \mathcal{G}_{\underline{s}} \text{ for all } \underline{s} \in \mathbb{R}_{\geq 0}^{md}\}.$$

Why only m terms?

$$\begin{pmatrix} V_t \\ Y_t \end{pmatrix} = \begin{pmatrix} v \\ y \end{pmatrix} + Z_t^{(0)} + Z^{(1)}_{\int_0^t V_s ds} + Z^{(2)}_{\int_0^t Y_s ds} .$$

Why only m terms?

$$\begin{pmatrix} V_t \\ Y_t \end{pmatrix} = \begin{pmatrix} v \\ y \end{pmatrix} + Z_t^{(0)} + Z^{(1)}_{\int_0^t V_s ds} + Z^{(2)}_{\int_0^t Y_s ds} .$$

Apply the ∞ transformation.

Why only m terms?

$$\begin{pmatrix} V_t \\ Y_t \end{pmatrix} = \begin{pmatrix} v \\ y \end{pmatrix} + Z^{(1)}_{\int_0^t V_s ds} + Z^{(2)}_{\int_0^t Y_s ds} .$$

Apply the method of the moving frame, see in [KST11].

Why only m terms?

$$\begin{pmatrix} V_t \\ Y_t \end{pmatrix} = \begin{pmatrix} v \\ y \end{pmatrix} + Z^{(1)}_{\int_0^t V_s ds} \, .$$

Why only m terms?

$$\begin{pmatrix} V_t \\ Y_t \end{pmatrix} = \begin{pmatrix} v \\ y \end{pmatrix} + Z^{(1)}_{\int_0^t V_s ds} .$$

Question: Given a Lévy process $Z^{(1)}$ taking values in $\mathbb{R}^2$, is there a solution of

$$\begin{cases} V_t &= v + Z_1^{(1)}\left(\int_0^t V_s ds\right) \\ Y_t &= y + Z_2^{(1)}\left(\int_0^t V_s ds\right) \end{cases} \qquad ?$$

The one dimensional case

Question: Given a Lévy process Z taking values in $\mathbb{R}$, is there a solution of

$$X_t = x + Z_{\int_0^t X_s ds} \qquad ?$$

- For the one dimensional case see also [Caballero et al., 2009, Caballero et al., 2013].

An ODE point of view in dimension 1

Introduce

$$\tau(t) := \int_0^t X_s ds \,.$$

Does there exist a solution of

$$\begin{cases} \dot{\tau}(t) = x + Z(\tau(t)) \\ \tau(0) = 0 \end{cases} \qquad ?$$

Focus on $\mathbb{R}^m_{\geq 0}$

Lemma

If

$$\begin{cases} \dot{\tau}_i(t) = x_i + \sum_{k=1}^{m} Z_i^{(k)} \left(\int_0^t X_r^{(k)} dr \right) \\ \tau_i(0) = 0 \end{cases}$$

admits a solution for all $i = 1, \dots, m$, then it admits also a solution for all $i = 1, \dots, d$.

An ODE point of view in $\mathbb{R}^m_{\geq 0}$

Introduce

$$\begin{aligned} \mathcal{Z} : \mathbb{R}^m_{\geq 0} & \rightarrow \mathbb{R}^m \\ \underline{s} & \mapsto \sum_{i=1}^{m} Z^{(i)}(s_i)\,. \end{aligned} \tag{1}$$

Does there exist a solution $\underline{\tau} \in \mathbb{R}^m_{\geq 0}$ of

$$\begin{cases} \dot{\underline{\tau}}(t) = x + \mathcal{Z}(\underline{\tau}(t))\,, \\ \underline{\tau}(0) = 0\,. \end{cases} \qquad ?$$

Decomposition of $\mathcal{Z}$

The Lévy–Itô decomposition together with the **canonical form** of the admissible parameters give

$$Z_t^{(i)} = \beta_i t + \sigma_i B_t^{(i)} + \int_0^t \int \xi 1_{\{|\xi|>1\}} \mathcal{J}^{(i)}(d\xi, ds) + \int_0^t \int \xi 1_{\{|\xi|\leq 1\}} (\mathcal{J}^{(i)}(d\xi, ds) - M_i(d\xi)ds)$$

where $\sigma_i = \sqrt{(\alpha_i)_{ii}}$, $B^{(i)}$ is a process in $\mathbb{R}^m$ which evolves only along the the i-th coordinate as Brownian motion and $\mathcal{J}^{(i)}$ is the jump measure of the process $Z^{(i)}$.

- Decompose

$$Z^{(i)} =: \widetilde{Z}^{(i)} + \approx{Z}^{(i)}$$

where $\widetilde{Z}^{(i)}$ and $\overset{\approx}{Z}^{(i)}$ are two stochastic processes on $\mathbb{R}^m$ defined by

$$\begin{aligned}
\widetilde{Z}_i^{(i)}(t) &:= (\beta_i)_i t + \sigma_i B^{(i)}(t) + \int_0^t \int \xi_i 1_{\{|\xi|>1\}} \mathcal{J}^{(i)}(d\xi, ds) \\
&\qquad + \int_0^t \int \xi_i 1_{\{|\xi|\leq 1\}} \widetilde{\mathcal{J}}^{(i)}(d\xi, ds) \\
\widetilde{Z}_k^{(i)}(t) &:= 0, \qquad \text{for } k \neq i\,,
\end{aligned}$$

$$\begin{aligned}
\overset{\approx}{Z}^{(i)}(t) &:= \overset{\approx}{\beta}_i t + \int_0^t \int (\xi - \xi_i e_i) 1_{\{|\xi|>1\}} \mathcal{J}^{(i)}(d\xi, ds) \\
&\qquad + \int_0^t \int (\xi - \xi_i e_i) 1_{\{|\xi|\leq 1\}} \widetilde{\mathcal{J}}^{(i)}(d\xi, ds)\,.
\end{aligned}$$

where $\overset{\approx}{\beta}_i = \beta_i - e_i(\beta_i)_i$ and $\widetilde{\mathcal{J}}^{(i)}$ is the compensated jump measure.

Approximation of the jump part

- Introduce, for all $\underline{s} \in \mathbb{R}^m_{\geq 0}$,

$$\tilde{\mathcal{Z}}(\underline{s}) := \sum_{i=1}^{m} \tilde{Z}^{(i)}(s_i), \quad \approx{\mathcal{Z}}(\underline{s}) := \sum_{i=1}^{m} \approx{Z}^{(i)}(s_i)\,.$$

- Fix $M \in \mathbb{N}$ and consider the partition

$$\mathcal{T}_M := \left\{ \frac{k}{2^M}, \quad k \geq 0 \right\}.$$

- Define the following approximations on the partition $\mathcal{T}_M$:

$$\uparrow\approx{Z}^{(i,M)}_t := \sum_{k=0}^{\infty} \approx{Z}^{(i)}_{k/2^M} 1_{[\frac{k}{2^M}, \frac{k+1}{2^M})}(t)\,,$$

$$\uparrow\approx{\mathcal{Z}}^{(M)}(\underline{s}) := \sum_{i=1}^{m} \uparrow\approx{Z}^{(i,M)}(s_i),$$

Construction of the time–change process

Theorem [G. and Teichmann, 2014b]

There exists a solution of

$$\begin{cases} \dot{\underline{\tau}}((t_0,\tau_0,x);t) = (x+\mathcal{Z})(\underline{\tau}((t_0,\tau_0,x);t)), \\ \underline{\tau}((t_0,\tau_0,x);t_0) = \tau_0 \,, \end{cases}$$

for $t \geq t_0$ and $\tau_0 \in \mathbb{R}^m_{\geq 0}$.

Construction of the time–change process

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$$\begin{cases} \dot{\underline{\tau}}((t_0, \tau_0, \mathbf{x}); t) = (\mathbf{x} + \mathcal{Z})(\underline{\tau}((t_0, \tau_0, \mathbf{x}); t)), \\ \underline{\tau}((t_0, \tau_0, \mathbf{x}); t_0) = \tau_0 \,, \end{cases}$$

for $t \geq t_0$ and $\tau_0 \in \mathbb{R}^m_{\geq 0}$.

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for $t \geq t_0$ and $\tau_0 \in \mathbb{R}^m_{\geq 0}$.

The proof

step 1 Decompose $x + \mathcal{Z} = x + \tilde{\mathcal{Z}} + \tilde{\tilde{\mathcal{Z}}}$

step 2 Approximate $x + \tilde{\mathcal{Z}} + \tilde{\tilde{\mathcal{Z}}} \sim x + \tilde{\mathcal{Z}} + {}^{\uparrow}\tilde{\tilde{\mathcal{Z}}}^{(M)}$

Solution of the decoupled system

Theorem (Theorem VI.1.1 in [Ethier and Kurtz, 1986])

There exists a solution of

$$\begin{cases} \dot{\underline{\tau}}((t_0, \tau_0, x); t) = (x + \widetilde{\mathcal{Z}})(\underline{\tau}((t_0, \tau_0, x); t)), \\ \underline{\tau}((t_0, \tau_0, x); t_0) = \tau_0 \, , \end{cases}$$

with $\tau_0 \in \mathbb{R}^m_{\geq 0}$.

The proof

- Set

$$(t_0, \tau_0, x) := (0, 0, x),$$
$$\overleftarrow{\sigma} := (0, \dots, 0),$$
$$\overrightarrow{\sigma} := (\sigma_1^{(1,M)}, \dots, \sigma_1^{(i,M)}, \dots, \sigma_1^{(m,M)})$$

- Solve

$$\begin{cases} \dot{\underline{\tau}}((0,0,x);t) = (x + \widetilde{\mathcal{Z}})(\underline{\tau}((0,0,x);t)), \\ \underline{\tau}((0,0,x);t_0) = \tau_0 \,, \end{cases}$$

for $t \in [0, t_1]$ where

$$t_1 := \sup\{t > 0 \mid \underline{\tau}((t_0, \tau_0, x); t) \leq \overrightarrow{\sigma}\}\,.$$

Remark There might be one or more indices i^*, where equality holds. Collect them in a set $I^* \subseteq \{1, \dots, m\}$.

- Update the values

$$\pi_{I^*}\overleftarrow{\sigma} := \pi_{I^*}\overrightarrow{\sigma},$$
$$\pi_{I^*}\overrightarrow{\sigma} := \pi_{I^*}\overrightarrow{\sigma}_{++},$$

where $\overrightarrow{\sigma}_{++}$ contains the next jumps of ${}^{\uparrow}\widetilde{\widetilde{Z}}^{(i,M)}$ for all $i \in I^*$ after $\overrightarrow{\sigma}_i$.

- Define

$$\tau_1 := \underline{\tau}((t_0, \tau_0, x); t_1)$$
$$x_1 := x + \Delta^{\uparrow}\widetilde{\widetilde{\mathcal{Z}}}^{(M)}(\overleftarrow{\sigma}).$$

- Solve

$$\begin{cases} \dot{\underline{\tau}}((t_1, \tau_1, x_1); t) = (x_1 + \widetilde{\widetilde{\mathcal{Z}}})(\underline{\tau}((t_1, \tau_1, x_1); t)), \\ \underline{\tau}((t_1, \tau_1, x_1); t_1) = \tau_1 , \end{cases}$$

for $t \in [t_1, t_2]$ where

$$t_1 := \sup\{t > t_1 \mid \underline{\tau}((t_1, \tau_1, x_1); t) \leq \overrightarrow{\sigma}\}\,.$$

- Define iteratively, for all $n \geq 1$

$$
\begin{aligned}
t_{n+1} &:= \sup\{t > 0 \mid \underline{\tau}((t_n, \tau_n, x_n); t) \leq \overrightarrow{\sigma}\}, \\
\tau_{n+1} &:= \underline{\tau}((t_n, \tau_n, x_n); t_{n+1}), \\
x_{n+1} &:= x_n + \Delta^{\uparrow} \tilde{\mathcal{Z}}^{(M)}(\overleftarrow{\sigma}),
\end{aligned}
$$

where, at each step $\overleftarrow{\sigma}$ and $\overrightarrow{\sigma}$ are updated.

Solution of the approximated problem

Theorem

There exists a solution of

$$\begin{cases} \dot{\underline{\tau}}^{(M)}((0,0,x);t) = (x + \tilde{\mathcal{Z}} + {}^{\uparrow}\tilde{\tilde{\mathcal{Z}}}^{(M)})(\underline{\tau}^{(M)}((0,0,x);t)), \\ \underline{\tau}^{(M)}((0,0,x);t_0) = 0\,. \end{cases}$$

Moreover it holds

$$\lim_{M\to\infty} \underline{\tau}^{(M)}((t_0,\tau_0,x);t) = \underline{\tau}((t_0,\tau_0,x);t)$$

where $\underline{\tau}$ solves

$$\begin{cases} \dot{\underline{\tau}}((t_0,\tau_0,x);t) = (x + \mathcal{Z})(\underline{\tau}((t_0,\tau_0,x);t)), \\ \underline{\tau}((t_0,\tau_0,x);t_0) = \tau_0\,. \end{cases}$$

Outline

Application: the Feller diffusion with jumps

The problem

given $Z_t = bt + B_t + \sum_{i=1}^{N_t} \mathbb{e}_i$,

where

- B is a Brownian motion,
- N is a Poisson process with rate λ,
- $\mathbb{e}_i$ are i.i.d.$\sim \mathcal{E}xp(1)$,

construct $X = (X_t)_{t \geq 0}$ such that

$$X = x + Z_{\int_0^{\cdot} X_r dr} .$$

Application: the Feller diffusion with jumps

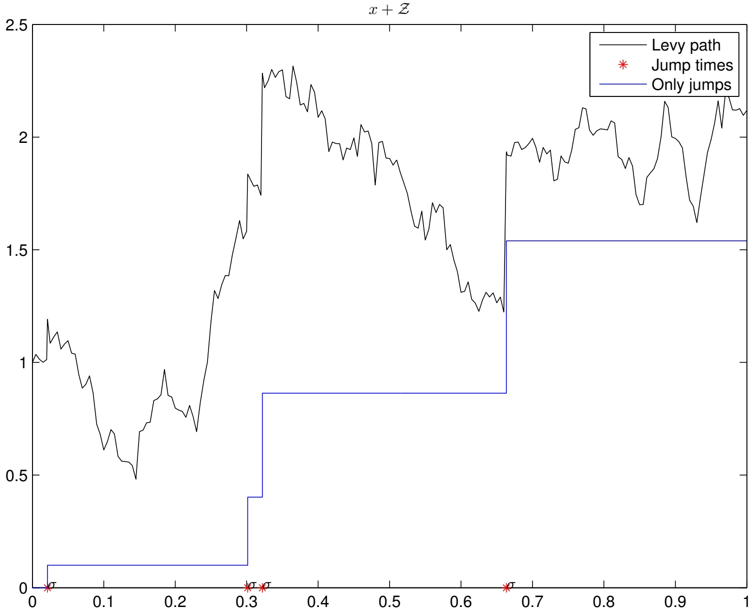
Observations

- If $\mathbf{Z_t = bt + B_t}$, the solution of the time–change equation is an affine process that starts at x with

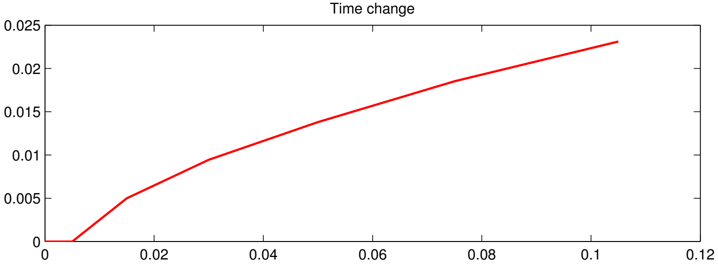
$$F(u) = 0, \quad R(u) = \frac{1}{2}u^2 + bu, \quad u \in \mathcal{U}\,.$$

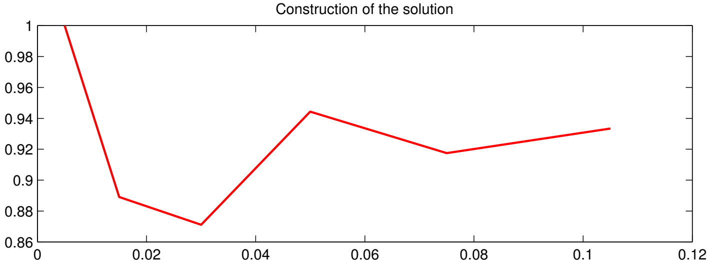
- Exact simulation or splitting schemes are available.
- Decompose the paths

$$Z_t = bt + B_t + \sum_{i=1}^{N_t} \mathbb{e}_i = \tilde{Z}_t + \overset{\approx}{Z}_t\,.$$

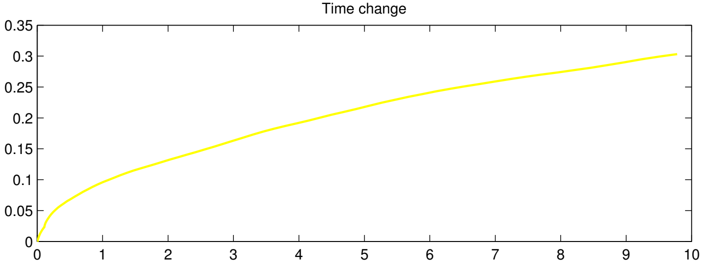

$x + \mathcal{Z}$
2.5
2
1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Levy path
Jump times
Only jumps
σ
σ
σ
σ

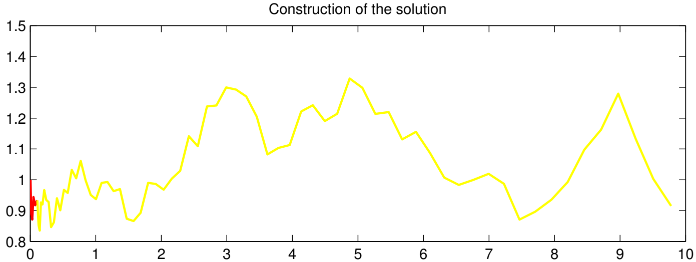
Application: the Feller diffusion with jumps



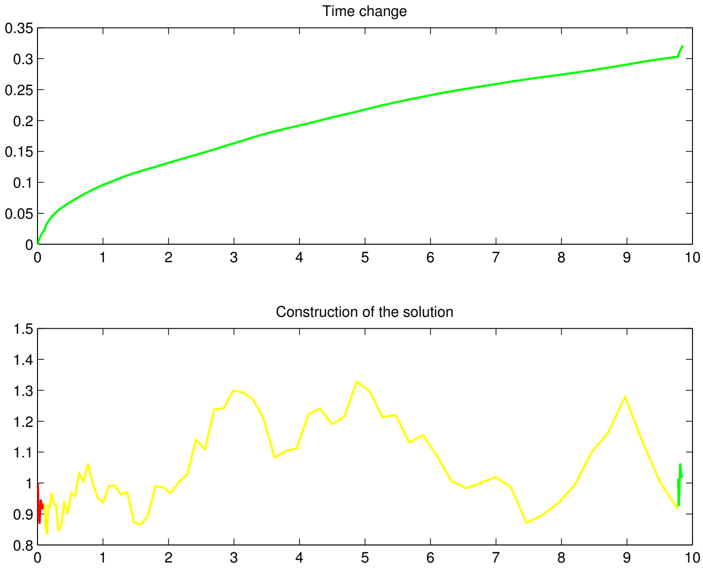


Application: the Feller diffusion with jumps

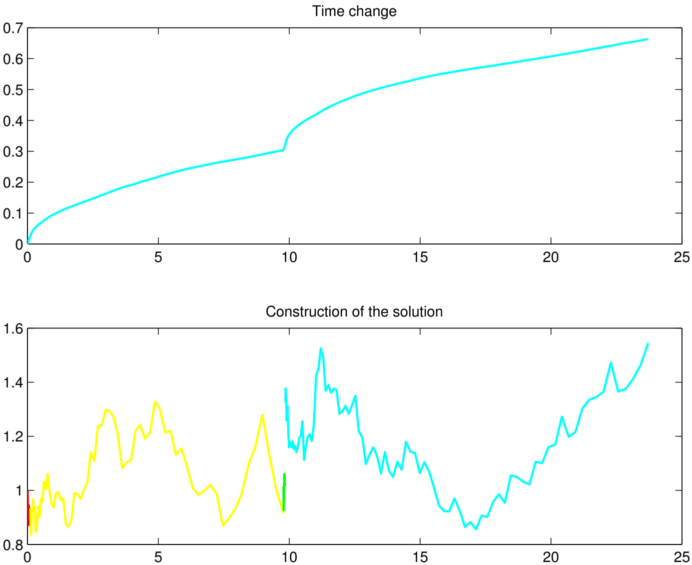




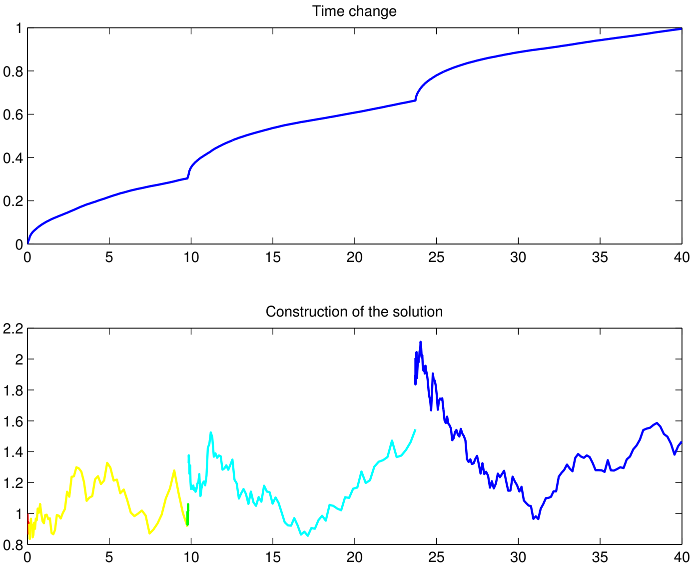
Application: the Feller diffusion with jumps



Application: the Feller diffusion with jumps



Application: the Feller diffusion with jumps



Thank you for your attention

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