Affine Processes from a different Perspective

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Introduction

The Problem

given

- $(X_t^x)_{t\in[0,T]}$ underlying stock
- $f(X^x)$ path dependent option depending on the whole path up to time T

find $\mathbb{E}^x\Big[f(X)\Big]$.

PDE Method

Solve numerically the pricing PDE

Probabilistic Method

Discretize the path and integrate by Monte Carlo methods

Numerical approximation of trajectories

High dimensional state space

$\hookrightarrow$ High order numerical schemes

Domain constrains

$\hookrightarrow$ Geometry preserving schemes

Example (A guiding example)

$$X_t^x := x + 2\int_0^t \sqrt{X_s}\, dW_s$$

$$L_x^t := 2t \sum_{k=1}^{N_{x/(2t)}} \mathbb{e}_k$$

$$X_t^x \overset{d}{=} L_x^t$$

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$$X^{x}{}_{t} \overset{d}{=} L_x^t$$

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Structure of the talk

Introduction

From Affine Processes to Lévy Processes and back

- A representation theorem
- An analytic approach

Applications

- Path approximation by time-space transformations
- CIR with jumps

Conclusions

Outline

Affine processes on the canonical state space

An *affine processes* on the canonical state space $D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ is a time homogeneous Markov process

$$X = (\Omega, (\mathcal{F}_t)_{t\geq 0}, (p_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{P}^x)_{x\in D_\Delta})$$

satisfying the following properties:

- (*stochastic continuity*) for every $t \geq 0$ and $x \in D$, $\lim_{s\to t} p_s(x, \cdot) = p_t(x, \cdot)$ weakly,
- (*affine property*) there exist functions $\phi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}$ and $\Psi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}^d$ such that

$$\mathbb{E}^x \left[e^{\langle u, X_t \rangle} \right] = \int_D e^{\langle u, \xi \rangle} p_t(x, d\xi) = e^{\phi(t,u) + \langle \Psi(t,u), x \rangle}$$

  for all $x \in D$ and $(t, u) \in \mathbb{R}_{\geq 0} \times \mathcal{U}$, with

$$\mathcal{U} = \left\{ u \in \mathbb{C}^d \mid e^{\langle u, x \rangle} \text{ is a bounded function on } D \right\}. \quad (1)$$

From affine processes to linear processes

Let $AP(D)$ be the space of affine processes with state space D. Define the map

$$\begin{aligned} \infty : AP(\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n) &\to AP(\mathbb{R}^{m+1}_{\geq 0} \times \mathbb{R}^n) \\ X &\mapsto X^{\infty} \end{aligned}$$

Input: X with $\mathbb{E}^x\left[e^{\langle u, X_t \rangle}\right] = e^{\phi(t,u) + \langle x, \Psi(t,u) \rangle}$

Output: X^{∞} with $\mathbb{E}^{(1,x)}\left[e^{\left\langle u, X_t^{\infty} \right\rangle}\right] = e^{\langle (1,x), \Psi^{\infty}(t,u) \rangle}$ where

$$\Psi^{\infty}(t, u_0, u_1, \ldots, u_d) := \begin{pmatrix} \phi(t, u_1, \ldots, u_d) + u_0 \\ \Psi(t, u_1, \ldots, u_d) \end{pmatrix}$$

Linear structure

There exists a function $\Psi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}^d$ such that

$$\mathbb{E}^x\left[e^{\langle u, X_t \rangle}\right] = \int_D e^{\langle u, \xi \rangle} p_t(x, d\xi) = e^{\langle \Psi(t,u), x \rangle}.$$

Generalized Riccati equations

On the set $\mathcal{Q} = \mathbb{R}_{\geq 0} \times \mathcal{U}$, the function Ψ satisfies the following system of generalized Riccati equations:

Generalized Riccati equations

$$\partial_t \Psi(t,u) \quad = \mathbf{R}(\Psi(t,u)), \quad \Psi(0,u) = u,$$

where for each $k = 1, \dots, d$ the function $\mathbf{R}_k$ has the following Lévy-Khintchine form

$$\begin{aligned} \mathbf{R}_k(u) \quad &= \quad \langle \beta_k, u \rangle + \frac{1}{2} \langle u, \alpha_k u \rangle - \gamma_k \\ &\quad + \int_{D \setminus \{0\}} \left(e^{\langle u, \xi \rangle} - 1 - \left\langle \pi_{J \cup \{k\}} u, \pi_{J \cup \{k\}} h(\xi) \right\rangle \right) M_k(d\xi). \end{aligned}$$

Infinite divisibility in D

Let $\mathcal{C}$ the *convex cone* of continuous function $\eta : \mathcal{U} \to \mathbb{C}_+$ of type

$$\eta(u) = \langle \mathrm{b}, u \rangle + \frac{1}{2} \langle \pi_J u, \sigma \pi_J u \rangle + \int_{D \setminus \{0\}} \left(e^{\langle u, \xi \rangle} - 1 - \langle \pi_J u, \pi_J h(\xi) \rangle \right) \nu(d\xi). \qquad (*)$$

Definition

A distribution λ on D_Δ is *infinitely divisible* if and only if its Laplace transform takes the form $e^{\eta(u) - \mathrm{c}}$, where η has the form $(*)$ and $\mathrm{c} = \log \lambda(D)$.

$p_t(x, \cdot)$ is infinitely divisible in D!

Lévy Khintchine decomposition of $\langle x, \Psi(t,u) \rangle$

For $i = 1, \dots, m$

$$\Psi_i(t,u) = \langle \mathrm{b_i(t)}, u \rangle + \frac{1}{2} \langle \pi_J u, \sigma_i(t) \pi_J u \rangle - \mathrm{c}_i(t) + \int_{D \setminus \{0\}} \left(e^{\langle u, \xi \rangle} - 1 - \langle \pi_J u, \pi_J h(\xi) \rangle \right) \nu_i(t, d\xi)$$

For $j = m+1, \dots, d$

$$\Psi_j(t,u) = \left\langle \mathrm{b_j(t)}, u \right\rangle .$$

Lévy Khintchine decomposition of $\langle x, \mathbf{R}(u) \rangle$

For $i = 1, \ldots, m$

$$\mathbf{R}_i(u) = \langle \beta_i, u \rangle + \frac{1}{2} \left\langle \pi_J u, \alpha^i_J \pi_J u \right\rangle + \frac{1}{2} \alpha^2_{i,i} u_i^2 - \gamma_i + \int_{D \setminus \{0\}} \left(e^{\langle u, \xi \rangle} - 1 - \left\langle \pi_{J \cup \{i\}} u, \pi_{J \cup \{i\}} h(\xi) \right\rangle \right) M_i(d\xi)$$

For $j = m+1, \ldots, d$

$$\mathbf{R}_j(u) = \left\langle \beta_j, u \right\rangle .$$

Outline

From affine processes to Lévy processes

Let $(\mathcal{D}(\mathbb{R}_{\geq 0}; D_\Delta), (\mathcal{F}_t)_{t\geq 0}, (p_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{P}^x)_{x\in D_\Delta})$ be a linear process on the canonical state space $D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$.

Proposition

For each fixed $t > 0$ and $x \in D \setminus \{0, \Delta\}$, there exists process $(L^t_{sx})_{s\in[0,1]}$ such that:

1. $L^t_0 = 0$,
2. for every $0 \leq s_1 \leq s_2 \leq 1$, the increment $L^t_{s_2 x} - L^t_{s_1 x}$ is independent of the family $(L^t_{sx})_{s\in[0,s_1]}$ and it distributed as $X_t^{(s_2 - s_1)x}$.

Moreover for any fixed $t \geq 0$, and $x \in D$ there exists a unique modification $\widetilde{L}^t$ of L^t which is a Lévy process with càdlàg paths.

The proof

Apply Kolmogorov's existence Theorem with the convolution semigroup $(p_t(sx,\cdot))_{s\geq 0}$.

Chapman-Kolmogorov's equations $\rightarrow$ Semigroup property in time

$p_{s+t}(x,\cdot) = p_s \cdot p_t(x,\cdot) := \int p_t(x,dy)p_s(y,\cdot)$,

Linearity $\rightarrow$ Convolution property in space

$p_t(x+y,\cdot) = p_t(x,\cdot) * p_t(y,\cdot)$.

Let t run

Remark: Here t appears here as a parameter of $(L^t_{sx})_{s\in[0,1]}$.

Idea: Let t evolve and consider the above construction on the path space.

Result: Construct a path valued process $(L^{\cdot}_{sx})_{s\in[0,1]}$ which starts in zero and it reaches $(X^x_t)_{t\geq 0}$ at time 1.

Make things rigorous

Theorem

There exists a process $(L_{sx})_{s\geq 0}$ *taking values in* $\mathcal{D}(\mathbb{R}_{\geq 0}; D_{\Delta})$ *such that*

1. *it has stationary and independent increments,*
2. *it is stochastically continuous,*
3. *it holds*

$$\mathbb{E}^{x}\left[e^{\langle u, X_t\rangle}\right] = e^{\langle x, \Psi(t,u)\rangle} = \mathbb{E}\left[e^{\langle u, \mathrm{ev}_t(L_{sx})\rangle}\right]\Big|_{s=1}$$

Proof: Apply Kolmogorov's existence Theorem with the convolution semigroup $\wp_s(x,\cdot) := \mathbb{P}^{sx},\ s \geq 0.$

Different scenarios

step 1 Solve $\Psi(t,u) = u + \int_0^t \mathbf{R}(\Psi(s,u))ds, \quad u \in \mathcal{U}.$

step 2 Do Lévy-Khintchine decomposition of $\Psi(t,u)$

$\mathbf{R}$
(β, σ, M)

$\Psi(t,\cdot)$
$(\mathrm{b}(t), \sigma(t), \nu(t,\cdot))$

step 1 Find $R(u) := \lim_{t\to 0} \frac{\Psi(t,u)-u}{t}$

step 2 Do Lévy-Khintchine decomposition of $\mathbf{R}$

Insight into Bochner subordinator

Definition

A function $\Psi(t,u) : \mathbb{R}_{\geq 0} \times \mathbb{C}^d \to \mathbb{C}^d$ is a *ray* if

1. $\langle x, \Psi(t,u) \rangle \in \mathcal{C}$,
2. $\Psi(t,u)$ in analytic in u and jointly continuous in (t,u),
3. $\Psi(t,u)$ is differentiable in t and $\lim_{t\to 0} \Psi(t,u) = u$,
4. $\Psi(t+s,u) = \Psi(t, \Psi(s,u))$.

From Markov property, for any $s, t \geq 0$

$$\mathbb{E}^x\left[e^{\langle u, X_{t+s}\rangle}\right] = \mathbb{E}^x\left[e^{\langle X_s, \Psi(t,u)\rangle}\right] = e^{\langle x, \Psi(s, \Psi(t,u))\rangle}.$$

An analytic approach

Semiflow property

For any $N > 0$ and $t > 0$

$$\begin{aligned}\Psi(t,u) &= \Psi(h,u)^{\circ N} \\ &:= \Psi\left(h, \Psi\left(h, \ldots, \Psi\left(h,u\right)\right)\right), \quad h = \frac{t}{N}.\end{aligned}$$

In general if

$$\lim_{N\to\infty} \Psi(h,u)^{\circ N} =: \widetilde{\Psi}(t,u), \;\; u \in \mathring{\mathcal{U}} \quad \text{then} \quad \langle x, \widetilde{\Psi}(t,u)\rangle \in \mathcal{C}.$$

Outline

Approximation of rays

1. Fix $N > 0$, $T > 0$ and a partition $\{t_0 = 0, t_1 = h, \dots, t_N = T\}$ with $h = \frac{T}{N}$.
2. Let y_n be the approximation of $\Psi(t_n, u)$.

Example: Forward Euler's method

For small h,

$$\mathbb{E}^x\left[e^{\langle u, X_h\rangle}\right] = e^{\langle x,u\rangle + h\langle x, \mathrm{R}(u)\rangle}, \quad u \in \mathcal{U}.$$

X_t^x is infinitely divisible in $D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ but $\langle x, \mathbf{R}(u)\rangle$ has the Lévy-Khintchine representation in $\mathbb{R}^d$!

$\rightsquigarrow$ Seek methods that preserve positivity of the semiflow.

Geometry preserving schemes

Input $\Psi(t,\cdot) \in \mathcal{C}$

$$\begin{aligned} y_0(u) &= u \\ y_{n+1}(u) &= \Psi(h, y_n(u)) \qquad n = 0, \dots, N-1, \end{aligned}$$

with $\langle x, \Psi(h,\cdot)\rangle \in \mathcal{C}$ for all $x \in D$.

if Ψ is a ray: for any $n = 0, \dots, N$, $y_n(u) = \Psi(t_n, u)$.
Output: X^x affine process corresponding to L_x.

if not? $\lim_{N\to\infty} y_N(u) = \widetilde{\Psi}(T, u)$, where $\widetilde{\Psi}(T, u)$ solves the Riccati equation driven by $\mathbf{r}(u) = \partial_t \Psi(t, u)_{|_{t=0^+}}$.
Output: X^x with functional characteristic $\mathbf{r}$.

Example 1: Feller diffusion $\leftrightsquigarrow$ homographic ray

$$\begin{cases} dX_t = \sqrt{X_t}\, dW_t \\ X_0 = x \end{cases}$$

Affine process on $\mathbb{R}_{\geq 0}$ with functional characteristic $\mathbf{R}(u) = \frac{u^2}{2}$.

One step Forward Euler's method

$$\begin{aligned} \widehat{X}^x_0 &= x \\ \widehat{X}^x_{t_{k+1}} &= \widehat{X}^x_{t_k} + \mathcal{N}(0, hy)_{|y=\max(0,\widehat{X}^x_{t_k})}, \qquad n = 0, \ldots, N-1. \end{aligned}$$

where $\mathcal{N}(0, \sigma^2)$ is a normal random variable with mean 0 and variance σ^2.

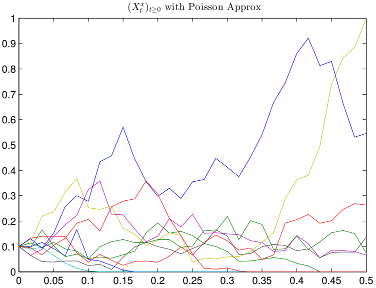
Example 1: Feller diffusion $\leftrightsquigarrow$ homographic ray

$$\mathbb{E}\left[e^{u\mathcal{L}_x^t}\right] = \exp\left(x\frac{u}{1-\frac{ut}{2}}\right), \quad \mathcal{R}e(u) < \frac{2}{t},$$

$\Psi^{\mathcal{L}}$ is a ray

$$\begin{aligned} \widehat{X}_0^x &= x, \\ \widehat{X}_{t_{k+1}}^x &= (\mathcal{L}_{sx}^h)_{s=1, x=\widehat{X}_{t_k}^x}, \; k \geq 0 \end{aligned}$$

where $\mathcal{L}_{\cdot x}^t$ is a subordinator with $\nu(t,x,d\xi) := \frac{4x}{t^2} e^{-\frac{2\xi}{t}} d\xi$.



Example 2: Neveu Branching $\leftrightsquigarrow$ stable ray

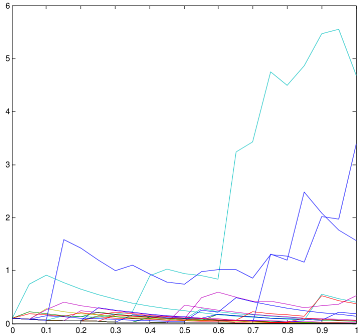
$$\mathbb{E}\left[e^{u\mathcal{J}_x^t}\right] = \exp\left(-x(-u)^{e^{-t}}\right)$$

Affine process on $\mathbb{R}_{\geq 0}$ with functional characteristic $\mathbf{R}(u) = -u\log(-u)$.

$\Psi^{\mathcal{J}}(t,u)$ is a ray

$$\begin{aligned} \widehat{X}_0^x &= x \\ \widehat{X}_{t_{k+1}}^x &= (\mathcal{J}_{sx}^h)_{s=1, x=\widehat{X}_{t_k}^x}, \ k \geq 0. \end{aligned}$$

where $\mathcal{J}_{\cdot x}^t$ is a subordinator with $\nu(t,x,d\xi) := \frac{xe^{-t}}{\Gamma(1-e^{-t})}\xi^{-1-e^{-t}}d\xi$.



A theorem for jump diffusions

Split

$$\mathbf{R}_i(u) = \langle \beta_i, u\rangle + \frac{1}{2}\langle u, \alpha_i u\rangle \qquad =: \mathbf{r}_i^1(u)$$

$$+ \int_D \left(e^{\langle u,\xi\rangle} - 1 - \left\langle \pi_{J\cup\{i\}} u, \pi_{J\cup\{i\}} h(\xi)\right\rangle \right) M_i(d\xi) \quad =: \mathbf{r}_i^2(u)$$

and consider

$$\partial_t \Psi^{\mathcal{L}}(t,u) = \mathbf{r}^1(\Psi^{\mathcal{L}}(t,u)), \qquad \partial_t \Psi^{\mathcal{J}}(t,u) = \mathbf{r}^2(\Psi^{\mathcal{J}}(t,u)).$$

Theorem

Let X^x be a conservative affine process. The scheme

$$\widehat{X}_0^x = x,$$
$$\widehat{X}_{t_{k+1}}^x = (\mathcal{L}^h \circ \mathcal{J}^h)_{\widehat{X}_{t_k}^x},$$

is a weak first order approximation for X^x.

Approximation of the CIR with jumps

The model

$$\begin{aligned} X_t^x &= x + \int_0^t cX_s ds + \int_0^t \sqrt{X_s} dW_s \\ &+ \int_0^t \int_D h(\xi) \left(N(ds, d\xi; X) - \frac{d\xi}{\xi^2} X_s ds \right) \\ &+ \int_0^t \int_D (\xi - h(\xi)) \, N(ds, d\xi; X). \end{aligned}$$

$$\mathbf{R}(u) = \frac{1}{2}u^2 + cu + \int_0^\infty \left(e^{u\xi} - 1 - uh(\xi) \right) \frac{d\xi}{\xi^2} =: \mathbf{r}^1(u) + \mathbf{r}^2(u).$$

Remark: $\Psi(h, u) := \Psi^{\mathcal{J}}(h, \Psi^{\mathcal{L}}(h, u))$ is NOT a ray but

$$\partial_t \Psi(t, u)_{|_{t=0^+}} = \mathbf{r}^1(u) + \mathbf{r}^2(u) = \mathbf{R}(u).$$

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Conclusions

Conclusions and future research

1. The understanding of linear processes as path valued Lévy processes leads to numerical schemes based on the approximation of **'easy to simulate'** processes.
2. **Higher order schemes** can be derived by performing a Strang splitting instead of the Lie-Trotter splitting.
3. The method is **flexible** since it relies on the concept of Bochner subordination.
4. This perspective allows us to consider **path dependent options** written on a stock driven by an Affine process as a European style option written on a path valued process.

Thank you