Regularity Results for Degenerate Kolmogorov Equation of Affine Type

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Introduction and motivations

Affine processes
- Examples and definitions
- Two key properties

The Result
- Regularity on C^{∞}_{pol}
- Sketch of the proof

Outline

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Set up

The state space: $D \subset \mathbb{R}^d$

The model: $X = (\Omega, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{P}^x)_{x\in D_\Delta})$ *càdlàg*, time homogeneous, *conservative* Markov process in D

$$p_t(x, A) = \mathbb{P}^x(X_t \in A), \qquad t \geq 0,\ x \in D,\ A \in \mathcal{B}(D).$$

The problem: Compute

$$P_t f(x) := \mathbb{E}^x\Big[f(X_t)\Big],$$

$t \geq 0,\ x \in D,\ f \in \mathcal{M}.$

Option Pricing

The problem:

given

- $(X_t)_{t\in[0,T]}$ stock process
- H payoff function, possibly depending on the whole path up to time T

find $\mathbb{E}^x\Big[H(X_t,\ t\in[0,T])\Big]$.

Approximation of trajectories by MC methods

step 1 Fix a uniform partition in $[0, T]$

$$\{t_0 = 0, \dots, t_k = kh, \dots, t_N = T\}, \quad h = \frac{T}{N}.$$

step 2 Find a piecewise constant approximating process $(\widehat{X}_{t_k})_{k=0,\dots,N}$ such that

- $\widehat{X}^x_{t_0} = x$,
- $\widehat{X}^x$ is a weak $\nu-$order approximation of X^x.

Definition

For every $f \in C^\infty$ with compact support there exists a $K > 0$ such that

$$\left| \mathbb{E}^x \left[f(X_T) \right] - \mathbb{E} \left[f(\widehat{X}^x_{t_N}) \right] \right| \leq K h^\nu.$$

Example

Heston model

Suppose $X = (V, Y)$ with

$$\begin{cases} V_t & = v + bt + \beta \int_0^t V_s ds + \varsigma \int_0^t \sqrt{V_s} dB_s^1 \\ Y_t & = y - \frac{1}{2} \int_0^t V_s ds + \int_0^t \sqrt{V_s} dB_s^2, \end{cases}$$

where

- $\beta, \varsigma \in \mathbb{R},\ b \in \mathbb{R}_{\geq 0}$,
- $B = (B^1, B^2)$ is a Brownian motion in $\mathbb{R}^2$ with correlation .

Approximation of the square root process

- Exact simulation
  - high computation time
  - need the knowledge of the exact distribution
- Euler scheme
  - not well defined
- NV splitting schemes [Ninomiya and Victoir, 2008]
  - do not rely on the specific model
  - high order convergence scheme
  - convergence holds under restrictions on the parameters
- Alfonsi scheme [Alfonsi, 2010]
  - extension of the NV schemes without restriction on the parameters

Convergence of weak schemes (see [Alfonsi, 2010])

The space $C^{\infty}_{pol}(D)$

$$C^{\infty}_{pol}(D) = \Big\{ f \in C^{\infty}(D), \text{ for all } \alpha \in \mathbb{N}^d \, \exists C_\alpha > 0, \eta_\alpha \in \mathbb{N} \mid \text{for all } x \in D \ |\partial^\alpha f(x)| \le C_\alpha (1 + |x|^{\eta_\alpha}) \Big\}.$$

Given $f \in C^{\infty}_{\mathrm{pol}}$ there exists $K(x,T) > 0$ such that

$$\Big| \mathbb{E}^x \Big[f(X_T) \Big] - \mathbb{E} \Big[f(\widehat{X}^x_{t_N}) \Big] \Big| \le K(x,T) h^{\nu}$$

if...

Moment condition

i) for all $h \in (0, h_0)$ and $\alpha \in \mathbb{N}$ there exists a constant C_α such that

$$\mathbb{E}\left[|\widehat{X}_h^x|^\alpha\right] \leq |x|^\alpha(1 + C_\alpha)h + C_\alpha h\,,$$

Short time approximation

ii) for all $h \in (0, h_0)$ there exist two constants $C, E > 0$ such that

$$\left|\mathbb{E}^x\left[f(X_h)\right] - \mathbb{E}\left[f(\widehat{X}_h^x)\right]\right| \leq C h^{\nu+1}(1 + |x|^E)\,,$$

Regularity of the Kolmogorov equation

iii) the function $u(t,x) := \mathbb{E}^x\left[f(X_t)\right]$ is well defined for $(t,x) \in [0,T] \times \mathbb{R}_{\geq 0}$, is a smooth solution of $\partial_t u(t,x) = \mathcal{A}u(t,x)$ such that, for all $\alpha \in \mathbb{N}^{d+1}$ multi–index it holds

$$\text{for all } (t,x) \in [0,T] \times \mathbb{R}_{\geq 0},\ |\partial^{\alpha}_{(t,x)} u(t,x)| \leq K(1+|x|^{\eta}),$$

where K and η are positive constants depending on the time horizon T and the order of derivative α.

Question: When is iii) satisfied?

[Talay and Tubaro, 1990] If $X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s$, with $b, \sigma \in C^\infty_{\mathrm{pol}}$.

[Alfonsi, 2005] For the CIR model.

[G. 2014] For affine processes

- Lévy processes
- The Heston model
- The Bates model

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Examples of affine processes 1

Lévy processes

$$Y_t = \boldsymbol{y} + \boldsymbol{\mu t} + \boldsymbol{\sigma B_t} + \int_0^t \int \xi 1_{\{|\xi| \leq 1\}} (J^Y(d\xi, ds) - m(d\xi) ds) + \int_0^t \int \xi 1_{\{|\xi| > 1\}} J^Y(d\xi, ds)$$

where (μ, α, m) is a Lévy triplet in $\mathbb{R}^n$, with $\alpha = \sigma\sigma^\top$.

Examples of affine processes 1

Lévy processes

$$
\begin{aligned}
Y_t =& \boldsymbol{y} + \boldsymbol{\mu t} + \boldsymbol{\sigma B_t} + \int_0^t \int \xi 1_{\{|\xi| \leq 1\}} (J^Y(d\xi, ds) - m(d\xi) ds) \\
&+ \int_0^t \int \xi 1_{\{|\xi| > 1\}} J^Y(d\xi, ds)
\end{aligned}
$$

where (μ, α, m) is a Lévy triplet in $\mathbb{R}^n$, with $\alpha = \sigma\sigma^\top$.

$$
\mathbb{E}^y \left[e^{\langle u, Y_t \rangle} \right] = e^{t\eta(u) + \langle y, u \rangle}, \qquad u \in \mathrm{i}\mathbb{R}^n .
$$

Examples of affine processes 2

Heston model

$$\begin{cases} V_t &= v + bt + \beta \int_0^t V_s ds + \varsigma \int_0^t \sqrt{V_s} dB_s^1 \\ Y_t &= y - \frac{1}{2} \int_0^t V_s ds + \int_0^t \sqrt{V_s} dB_s^2, \end{cases}$$

where

- $\beta, \varsigma \in \mathbb{R}$, $b \in \mathbb{R}_{\geq 0}$,
- $B = (B^1, B^2)$ is a Brownian motion in $\mathbb{R}^2$ with correlation .

Examples of affine processes 2

Heston model

$$\begin{cases} V_t &= v + bt + \beta \int_0^t V_s ds + \varsigma \int_0^t \sqrt{V_s} dB_s^1 \\ Y_t &= y - \frac{1}{2} \int_0^t V_s ds + \int_0^t \sqrt{V_s} dB_s^2, \end{cases}$$

where

- $\beta, \varsigma \in \mathbb{R},\ b \in \mathbb{R}_{\geq 0}$,
- $B = (B^1, B^2)$ is a Brownian motion in $\mathbb{R}^2$ with correlation .

$$\mathbb{E}^{(v,y)}\left[e^{u_1 V_t + u_2 Y_t}\right] = e^{\phi(t,u_1,u_2) + v\psi(t,u_1,u_2) + y u_2}, \quad (u_1, u_2) \in i\mathbb{R}^2 .$$

Examples of affine processes 3

Bates model

$$\begin{cases} V_t & = v + bt + \beta \int_0^t V_s ds + \varsigma \int_0^t \sqrt{V_s} dB_s^1 \\ Y_t & = y - \frac{1}{2} \int_0^t V_s ds + \int_0^t \sqrt{V_s} dB_s^2 + \boldsymbol{J_t}, \end{cases}$$

where

- J is a compound Poisson process

Examples of affine processes 3

Bates model

$$\begin{cases} V_t &= v + bt + \beta \int_0^t V_s ds + \varsigma \int_0^t \sqrt{V_s} dB_s^1 \\ Y_t &= y - \frac{1}{2} \int_0^t V_s ds + \int_0^t \sqrt{V_s} dB_s^2 + J_t, \end{cases}$$

where

- J is a compound Poisson process

$$\mathbb{E}^{(v,y)}\left[e^{u_1 V_t + u_2 Y_t}\right] = e^{\phi(t,u_1,u_2) + v\psi(t,u_1,u_2) + y u_2}, \quad (u_1, u_2) \in \mathrm{i}\mathbb{R}^2 .$$

In the above examples:

V stochastic variance process in $\mathbb{R}^m_{\geq 0}$,

Y (discounted) log–price process in $\mathbb{R}^n$,

and

- $X := (V, Y)$ is a time homogeneous Markov process in $D := \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$,
- there exist functions $\phi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}$ and $\Psi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}^d$ such that

$$\mathbb{E}^{(v,y)}\left[e^{\langle u_1, V_t\rangle + \langle u_2, Y_t\rangle}\right] = e^{\phi(t,u_1,u_2) + \langle v, \Psi(t,u_1,u_2)\rangle + \langle y, u_2\rangle},$$

for $u = (u_1, u_2) \in \mathcal{U}$, where $\mathcal{U} = \mathrm{i}\mathbb{R}^{m+n}$.

Let

$$(\Omega, (X_t)_{t\geq 0}, (\mathcal{F}_t)_{t\geq 0}, (p_t)_{t\geq 0}, (\mathbb{P}^x)_{x\in D})$$

be a time homogeneous Markov process. The process X is said to be an affine process if it satisfies the following properties:

- for every $t \geq 0$ and $x \in D$, $\lim_{s\to t} p_s(x, \cdot) = p_t(x, \cdot)$ weakly,
- there exist functions $\phi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}$ and $\Psi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}^d$ such that

$$\mathbb{E}^x\left[e^{\langle u, X_t\rangle}\right] = \int_D e^{\langle u,\xi\rangle} p_t(x, d\xi) = e^{\phi(t,u)+\langle x, \Psi(t,u)\rangle},$$

for all $x \in D$ and $(t, u) \in \mathbb{R}_{\geq 0} \times \mathcal{U}$, where $\mathcal{U} = \mathbb{C}^m_{\leq 0} \times \mathrm{i}\mathbb{R}^n$.

Assumption (A)

- X is conservative.
- There exists a function $\Psi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}^d$ such that

$$\mathbb{E}^x\left[e^{\langle u, X_t \rangle}\right] = \int_D e^{\langle u, \xi \rangle} p_t(x, d\xi) = e^{\langle x, \Psi(t,u) \rangle}.$$

  for all $x \in D$ and $(t, u) \in \mathbb{R}_{\geq 0} \times \mathcal{U}$.
- For all $y \in \mathbb{R}^d$ and $x \in D$,

$$\mathbb{E}^x\left[e^{\langle y, X_T \rangle}\right] < \infty$$

  for some fixed $T > 0$.

AP as polynomial processes

$\mathcal{P}_\eta$ vector space of polynomials up to degree $\eta \geq 0$

$$\mathcal{P}_\eta := \left\{ \mathbb{R}^d \ni x \mapsto \sum_{|k|=0}^{\eta} \alpha_k x^k \;\middle|\; \alpha_k \in \mathbb{R} \right\}$$

Theorem 2.14 in [Cuchiero et al., 2008]

Under the Assumption (A), X is a a polynomial process, i.e. for all $\eta \in \mathbb{N}$ and $f \in \mathcal{P}_\eta$

$$x \mapsto \mathbb{E}^x \Big[f(X_t) \Big] \in \mathcal{P}_\eta, \quad \text{for all } t \geq 0\,.$$

AP from the perspective of path-space valued LP

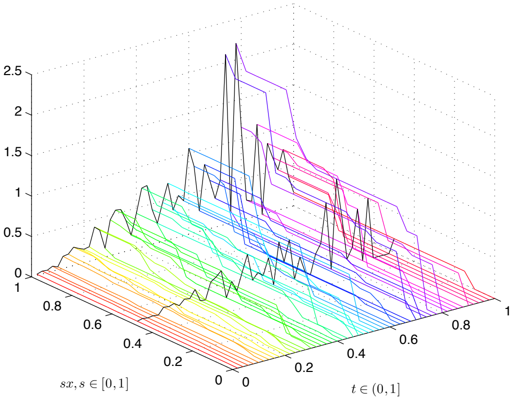
Proposition 4.1. in [G., Teichmann, 2014]

For each fixed $t > 0$ and $x \in D$, there exists a process $(L_s^{(t,x)})_{s \geq 0}$ such that:

1. $L_0^{(t,x)} = 0$,
2. for every $0 \leq s_1 \leq s_2 < \infty$, the increment $L_{s_2}^{(t,x)} - L_{s_1}^{(t,x)}$ is independent of the family $(L_s^{(t,x)})_{s \in [0,s_1]}$ and it is distributed as $X_t^{(s_2 - s_1)x}$,
3. it is stochastically continuous.

A different perspective

$$\mathbb{E}^x\left[e^{uX_t}\right] = e^{x\frac{u}{1-2ut}} = \mathbb{E}\left[e^{uL_1^{(t,x)}}\right]$$



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The result

Theorem [G., 2014]

Let $f \in C^{\infty}_{\mathrm{pol}}$. Then, under the Assumption (A) the function $u : \mathbb{R}_{\geq 0} \times D \to \mathbb{R}$ defined by $u(t,x) = \mathbb{E}^{x}\Big[f(X_t)\Big]$ is smooth, with all derivatives satisfying the following property:

$$\text{for all } (t,x) \in [0,T] \times D,\ |\partial^{\alpha}_{(t,x)} u(t,x)| \leq K_{\alpha}(T)(1 + |x|^{\eta_{\alpha}(T)}),$$

where $K_{\alpha}(T)$ and $\eta_{\alpha}(T)$ are positive constants depending on the time horizon T and the order of derivative α.

Dissect the theorem

part 1 $t \mapsto P_t f(x)$ is differentiable for all $x \in D$.

$\hookrightarrow$ Use iterated Dynkin formula.

part 2 $x \mapsto P_t f(x)$ is differentiable.

$\hookrightarrow$ Do a time-space shift.

part 3 $(t, x) \mapsto P_t f(x)$ is differentiable with controlled growth.

Theorem: Regularity in time

Under the Assumption (A) it holds

(i) for any $f \in C^{\infty}_{\mathrm{pol}}$, $P_t f$ solves the Kolmogorov's equation

$$\partial_t u(t,x) = \mathcal{A}u(t,x)\,,$$
$$u(0,x) = f(x)\,,$$

for $(t,x) \in [0,T] \times D$,

(ii) for any $f \in C^{\infty}_{\mathrm{pol}}$ and $\nu \in \mathbb{N}$ the following expansion of the transition semigroup holds for $(t,x) \in [0,T] \times D$:

$$\mathbb{E}^x\Big[f(X_t)\Big] = f(x) + \sum_{k=1}^{\nu} \frac{t^k}{k!}\mathcal{A}^k f(x) + \mathcal{R}_\nu f(x,t),$$

where $\mathcal{R}_\nu f(x,t)$ is a remainder of order $\mathcal{O}(t^{\nu+1})$.

Regularity in space

step 1 Consider the decomposition

$$X^{x+hx_i} \stackrel{law}{=} X^x + \widetilde{X}^{hx_i}, \; h > 0, \; i = 1, \ldots, d,$$

where $\widetilde{X}^{hx_i}$ is an independent copy of the process X starting from hx_i.

step 2 For fixed $(t, x) \in \mathbb{R}_{\geq 0} \times D$, $X_t^{x+hx_i}$ has the same distribution as the distribution of the Lévy process $L^{(t,x_i)}$ at time h starting from the initial random position X_t^x.

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Thank you