Lamperti transform for multi-type CBI

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Structure of the talk

1. Introduction
   - Real valued CB and Lamperti transform
   - Extension to real valued CBI
2. What is a multi–type CBI
   - Definitions
   - Some additional results
3. Lamperti transform for multi–type CBI
   - Multi–type CB and their time–change representation
   - Extension to multi–type CBI
   - Sketch of the proof

Outline

Branching processes

Let

$$(\Omega, (X_t)_{t\geq 0}, (\mathcal{F}_t^\natural)_{t\geq 0}, (\mathbb{P}^x)_{x\in\mathbb{R}_{\geq 0}})$$

be a time homogeneous Markov process. The process X is said to be an branching process if it satisfies the following property:

Branching property

For any $t \geq 0$ and $x_1, x_2 \in \mathbb{R}_{\geq 0}$, the law of X_t under $\mathbb{P}^{x_1+x_2}$ is the same as the law of $X_t^{(1)} + X_t^{(2)}$, where each $X^{(i)}$ has the same distribution as X under $\mathbb{P}^{x_i}$, for $i = 1, 2$.

Fourier–Laplace transform characterization

- There exists a function $\Psi : \mathbb{R}_{\geq 0} \times \mathbb{C}_{\leq 0} \to \mathbb{C}$ such that

$$\mathbb{E}^x \left[e^{uX_t} \right] = e^{x\Psi(t,u)},$$

  for all $x \in \mathbb{R}_{\geq 0}$ and $(t, u) \in \mathbb{R}_{\geq 0} \times \mathbb{C}_{\leq 0}$.

- On the set $\mathcal{Q} = \mathbb{R}_{\geq 0} \times \mathbb{C}_{\leq 0}$, the function Ψ satisfies the equation

$$\partial_t \Psi(t, u) \;\; = R(\Psi(t, u)), \quad \Psi(0, u) = u\,.$$

The branching mechanism

The function R has the following Lévy-Khintchine form

$$R(u) = \beta u + \frac{1}{2} u^2 \alpha + \int_0^\infty \left(e^{u\xi} - 1 - u\xi \mathbb{1}_{\{|\xi| \leq 1\}} \right) M(d\xi)\,,$$

where $\beta \in \mathbb{R}$, $\alpha \geq 0$ and M is a Lévy measure with support in $\mathbb{R}_{\geq 0}$.

Lévy processes

A time homogeneous Markov process Z is a Lévy process if the following three conditions are satisfied:

L1) $Z_0 = 0$ $\mathbb{P}$-a.s.

L2) Z has independent and stationary increments, i.e. for all $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < \ldots < t_{n+1} < \infty$

(independence) the random variables $\{Z_{t_{j+1}} - Z_{t_j}\}_{j=0,\ldots,n}$ are independent,

(stationarity) the distribution of $Z_{t_{j+1}} - Z_{t_j}$ coincides with the distribution of $Z_{(t_{j+1}-t_j)}$,

L3) (stochastic continuity) for each $a > 0$ and $s \geq 0$, $\lim_{t\to s} \mathbb{P}(|Z_t - Z_s| > a) = 0\,.$

Relation with infinitely divisible distributions

- If Z is a Lévy process, then, for any $t \geq 0$, the random variable Z_t is infinitely divisible.
- The Fourier transform of a Lévy process takes the form:

$$\mathbb{E}^0\left[e^{\langle u, Z_t\rangle}\right] = e^{t\eta(u)}, \quad u \in \mathrm{i}\mathbb{R}$$

$$\eta(u) = \beta u + \frac{1}{2}u^2\alpha + \int \left(e^{u\xi} - 1 - u\xi\mathbb{1}_{\{|\xi|\leq 1\}}\right) M(d\xi),$$

where $\beta \in \mathbb{R}$, $\alpha \geq 0$ and M is a Lévy measure in $\mathbb{R}$.

- The Fourier transform can be extended in the complex domain and the resulting Fourier–Laplace transform is well defined in

$$\mathcal{U} := \{u \in \mathbb{C} \mid \eta(\mathcal{R}e(u)) < \infty\}.$$

Lamperti transform

Theorem [Lamperti, 1967]

Let Z be a Lévy process with no negative jumps with Lévy exponent R, i.e.

$$\mathbb{E}^0\left[e^{uZ_t}\right] = e^{tR(u)}, \qquad u \in \mathcal{U}\,.$$

Define, for $t \geq 0$

$$X_t = x + Z_{\theta_t \wedge \tau_0^-} \quad \theta_t := \inf\left\{s > 0 \mid \int_0^s \frac{dr}{Z_r} > t\right\}.$$

Then X is a CB process with branching mechanism R.

Lamperti transform

Theorem 2 in [Caballero et al., 2013]

Let Z be a Lévy process with no negative jumps with Lévy exponent R, i.e.

$$\mathbb{E}^0\left[e^{uZ_t}\right] = e^{tR(u)}, \qquad u \in \mathcal{U}\,.$$

The time–change equation

$$X_t = x + Z_{\int_0^t X_r dr}$$

admits a unique solution, which is a CB process with branching mechanism R.

Outline

Add immigration

A CBI-process with branching mechanism R and immigration mechanism F is a Markov process Z taking values in $\mathbb{R}_{\geq 0}$ satisfying

- there exist functions $\phi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}$ and $\Psi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}$ such that
$$\mathbb{E}^x \left[e^{uX_t} \right] = e^{\phi(t,u) + x\Psi(t,u)},$$
for all $x \in \mathbb{R}_{\geq 0}$ and $(t, u) \in \mathbb{R}_{\geq 0} \times \mathcal{U}$.
- On the set $\mathcal{Q} = \mathbb{R}_{\geq 0} \times \mathcal{U}$, the functions ϕ and Ψ satisfy the following system :

$$\begin{aligned} \partial_t \phi(t,u) &= F(\Psi(t,u)), \quad \phi(0,u) = 0, \\ \partial_t \Psi(t,u) &= R(\Psi(t,u)), \quad \Psi(0,u) = u\,. \end{aligned}$$

Add immigration

The immigration mechanism

The function F has the following Lévy-Khintchine form

$$F(u) = bu + \int_0^\infty \left(e^{u\xi} - 1\right) m(d\xi)\,,$$

with $b \geq 0$ and m is a Lévy measure on $\mathbb{R}_{\geq 0}$ such that $\int (1 \wedge \xi) m(d\xi) < \infty$.

Lamperti transform for CBI

Theorem 2 in [Caballero et al., 2013]

Let $Z^{(1)}$ be a Lévy process with no negative jumps and $Z^{(0)}$ an independent subordinator such that

$$\mathbb{E}^0\left[e^{uZ_t^{(1)}}\right] = e^{tR(u)} \text{ and } \mathbb{E}^0\left[e^{uZ_t^{(0)}}\right] = e^{tF(u)}, \quad u \in \mathcal{U}.$$

The time–change equation

$$X_t = x + Z_t^{(0)} + Z^{(1)}_{\int_0^t X_r dr}$$

admits a unique solution, which is a CBI process with branching mechanism R and immigration mechanism F.

Outline

Definition

Let

$$(\Omega, (X_t)_{t\geq 0}, (\mathcal{F}_t^\natural)_{t\geq 0}, (\mathbb{P}^x)_{x\in\mathbb{R}^m_{\geq 0}})$$

be a time homogeneous Markov process. The process X is said to be a multi–type CBI if it satisfies the following property:

See [Duffie et al., 2003, Barczy et al., 2014]

There exist functions $\phi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}$ and $\Psi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}^m$ such that

$$\mathbb{E}^x \left[e^{\langle u, X_t \rangle} \right] = e^{\phi(t,u) + \langle x, \Psi(t,u) \rangle},$$

for all $x \in \mathbb{R}^m_{\geq 0}$ and $(t, u) \in \mathbb{R}_{\geq 0} \times \mathcal{U}$, with $\mathcal{U} = \mathbb{C}^m_{\leq 0}$.

Generalized Riccati equations

On the set $\mathcal{Q} = \mathbb{R}_{\geq 0} \times \mathcal{U}$, the functions ϕ and Ψ satisfy the following system of generalized Riccati equations:

$$\begin{aligned} \partial_t \phi(t,u) &= F(\Psi(t,u)), \quad \phi(0,u) = 0, \\ \partial_t \Psi(t,u) &= R(\Psi(t,u)), \quad \Psi(0,u) = u\,. \end{aligned}$$

Lévy–Khintchine form for the vector fields

The functions F and R_k, for each $k = 1, \ldots, m$, have the following Lévy-Khintchine form

$$F(u) = \langle b, u \rangle + \int_{\mathbb{R}^m_{\geq 0} \setminus \{0\}} \left(e^{\langle u, \xi \rangle} - 1 \right) m(d\xi),$$

$$R_k(u) = \langle \beta_k, u \rangle + \frac{1}{2} u_k^2 \alpha_k + \int_{\mathbb{R}^m_{\geq 0} \setminus \{0\}} \left(e^{\langle u, \xi \rangle} - 1 - \mathbf{u}_k \boldsymbol{\xi}_k \mathbb{1}_{\{|\xi| \leq 1\}} \right) M_k(d\xi).$$

Admissible parameters

The set of parameters satisfies the following restrictions

- $b, \beta_i \in \mathbb{R}^m$, $i = 1, \ldots, m$,
- $\alpha_i \geq 0$,
- m, M_i, $i = 1, \ldots, m$, Lévy measures.

drift	
$b \in \mathbb{R}^m_{\geq 0}$,	
$(\beta_i)_k \geq 0$,	for all $i = 1, \ldots, m$ and $k \neq i$,

jumps	
$\operatorname{supp} m \subseteq \mathbb{R}^m_{\geq 0}$,	and $\int (\lvert\xi\rvert \wedge 1)\, m(d\xi) < \infty$,
$\operatorname{supp} M_i \subseteq \mathbb{R}^m_{\geq 0}$,	for all $i = 1, \ldots, m$ and
	$\int \big((\lvert(\xi)_{-i}\rvert + \lvert\xi_i\rvert^2) \wedge 1\big) M_i(d\xi) < \infty$.

Outline

Remarks

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- It is possible to define $m+1$ independent Lévy processes $Z^{(0)}, Z^{(1)}, \dots, Z^{(m)}$ taking values in $\mathbb{R}^m$ with Lévy exponents $F, R_1, \dots, R_m$.

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- In [Kallsen, 2006] it has been proved that the time change equation

$$X_t = x + Z_t^{(0)} + \sum_{k=1}^{m} Z^{(k)}_{\int_0^t X_s^{(k)} ds}, \quad t \geq 0, \qquad (*)$$

admits a weak solution, i.e. there exists a probability space containing two processes (X, Z) such that $(*)$ holds in distribution.

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- Moreover X has the distribution of a multi–type CBI with immigration mechanism F and branching mechanism $(R_1, \dots, R_m)$.

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Does there exist a pathwise solution of $(*)$?

Outline

Results for multi–type CB

Theorem [G. and Teichmann, 2014]

Let $Z^{(1)}, \dots, Z^{(m)}$ be independent $\mathbb{R}^m$-valued Lévy processes with

$$\mathbb{E}^0\left[e^{\left\langle u, Z_t^{(k)} \right\rangle}\right] = e^{tR_k(u)}, \qquad u \in \mathcal{U}\,,$$

where each R_k is of LK form with triplets given by a set of admissible parameters. Then the time–change equation

$$X_t = x + \sum_{k=1}^{m} Z^{(k)}_{\int_0^t X_s^{(k)} ds} \quad t \geq 0\,,$$

admits a unique solution, which is a multi–type CB process with respect to the **time–changed filtration**.

Multiparameter time–change filtration

- Define
  $Z = (Z_1^{(1)}, \dots, Z_m^{(1)}, \dots, Z_1^{(m)}, \dots, Z_m^{(m)}) =: (Z^{(1)}, \dots, Z^{(m^2)})$.
- For all $\underline{s} = (s_1, \dots, s_{m^2}) \in \mathbb{R}_{\geq 0}^{m^2}$
  $\mathcal{G}_{\underline{s}}^{\natural} := \sigma\left(\{Z_{t_h}^{(h)},\ t_h \leq s_h,\ \text{for } h = 1, \dots, m^2\}\right)$.
- Complete it by $\mathcal{G}_{\underline{s}} = \bigcap_{n\in\mathbb{N}} \mathcal{G}^{\natural}_{\underline{s}^{(n)}+\frac{1}{n}} \vee \sigma(\mathcal{N})$.

Definition

A random variable $\underline{\tau} = (\tau_1, \dots, \tau_{m^2}) \in \mathbb{R}_{\geq 0}^{m^2}$ is a $(\mathcal{G}_{\underline{s}})$-stopping time if

$$\{\underline{\tau} \leq \underline{s}\} := \{\tau_1 \leq s_1, \dots, \tau_{m^2} \leq s_{m^2}\} \in \mathcal{G}_{\underline{s}}, \text{ for all } \underline{s} \in \mathbb{R}_{\geq 0}^{m^2}.$$

If $\underline{\tau}$ is a stopping time,

$$\mathcal{G}_{\underline{\tau}} := \{B \in \mathcal{G} \mid B \cap \{\underline{\tau} \leq \underline{s}\} \in \mathcal{G}_{\underline{s}} \text{ for all } \underline{s} \in \mathbb{R}_{\geq 0}^{m^2}\}.$$

Outline

Extension to multi–type CBI

- Let F be an immigration mechanism and $R = (R_1, \dots, R_m)$ a branching mechanism.
- Let $Z^{(0)}$ Lévy process with exponent F and $Z^{(i)}$ Lévy process with exponent R_i.
- Define, for $k = 0, \dots, m$,

$$\overline{Z}^{(k)} := \big(\overline{Z}_0^{(k)}, \underbrace{\overline{Z}_1^{(k)}, \dots, \overline{Z}_m^{(k)}}_{m \text{ coordinates}}\big) := \big(0, \underbrace{Z^{(k)}}_{m \text{ coordinates}}\big).$$

- Given $y = (1, x)$ with $x \in \mathbb{R}^m_{\geq 0}$, the previous result gives pathwise existence of

$$Y_t = y + \sum_{k=0}^{m} \overline{Z}^{(k)}_{\int_0^t Y_s^{(i)} ds}.$$

- It holds $Y = (1, X)$ where X is a $CBI(F, R)$.

Outline

The one dimensional case

Question: Given a Lévy process Z taking values in $\mathbb{R}$, is there a solution of

$$X_t = x + Z_{\int_0^t X_s ds} \qquad ?$$

- For the one dimensional case see also [Caballero et al., 2013].

An ODE point of view in $\mathbb{R}_{\geq 0}$

Introduce

$$\tau(t) := \int_0^t X_s ds \,.$$

Does there exist a solution $\tau \in \mathbb{R}_{\geq 0}$ of

$$\begin{cases} \dot{\tau}(t) = x + Z(\tau(t)) \\ \tau(0) = 0 \end{cases} \qquad ?$$

An ODE point of view in $\mathbb{R}^m_{\geq 0}$

Introduce

$$\begin{aligned} \mathcal{Z} : \mathbb{R}^m_{\geq 0} &\rightarrow \mathbb{R}^m \\ \underline{s} &\mapsto \sum_{i=1}^{m} Z^{(i)}(s_i)\,. \end{aligned}$$

Does there exist a solution $\underline{\tau} \in \mathbb{R}^m_{\geq 0}$ of

$$\begin{cases} \dot{\underline{\tau}}(t) = x + \mathcal{Z}(\underline{\tau}(t))\,, \\ \underline{\tau}(0) = 0\,. \end{cases} \qquad ?$$

Construction of the time–change process

Theorem [G. and Teichmann, 2014]

There exists a solution of

$$\begin{cases} \dot{\underline{\tau}}((t_0, \tau_0, x); t) = (x + \mathcal{Z})(\underline{\tau}((t_0, \tau_0, x); t)), \\ \underline{\tau}((t_0, \tau_0, x); t_0) = \tau_0 \, , \end{cases}$$

for $t \geq t_0$ and $\tau_0 \in \mathbb{R}^m_{\geq 0}$.

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Construction of the time–change process

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for $t \geq t_0$ and $\tau_0 \in \mathbb{R}^m_{\geq 0}$.

Decomposition of $\mathcal{Z}$

The Lévy–Itô decomposition together with the **canonical form** of the admissible parameters give

$$Z_t^{(i)} = \beta_i t + \sigma_i B_t^{(i)} + \int_0^t \int \xi 1_{\{|\xi|>1\}} \mathcal{J}^{(i)}(d\xi, ds) + \int_0^t \int \xi 1_{\{|\xi|\leq 1\}} (\mathcal{J}^{(i)}(d\xi, ds) - M_i(d\xi)ds)$$

where $\sigma_i = \sqrt{\alpha_i}$, $B^{(i)}$ is a process in $\mathbb{R}^m$ which evolves only along the the i-th coordinate as Brownian motion and $\mathcal{J}^{(i)}$ is the jump measure of the process $Z^{(i)}$.

- Decompose

$$Z^{(i)} =: \widetilde{Z}^{(i)} + \widetilde{\widetilde{Z}}^{(i)}$$

where $\widetilde{Z}^{(i)}$ and $\widetilde{\widetilde{Z}}^{(i)}$ are two stochastic processes on $\mathbb{R}^m$ defined by

$$\begin{aligned}
\widetilde{Z}_i^{(i)}(t) &:= (\beta_i)_i t + \sigma_i B^{(i)}(t) + \int_0^t \int \xi_i 1_{\{|\xi|>1\}} \mathcal{J}^{(i)}(d\xi, ds) \\
&\qquad + \int_0^t \int \xi_i 1_{\{|\xi|\leq 1\}} \widetilde{\mathcal{J}}^{(i)}(d\xi, ds) \\
\widetilde{Z}_k^{(i)}(t) &:= 0, \qquad \text{for } k \neq i\,,
\end{aligned}$$

$$\begin{aligned}
\widetilde{\widetilde{Z}}^{(i)}(t) &:= \widetilde{\widetilde{\beta}}_i t + \int_0^t \int (\xi - \xi_i e_i) 1_{\{|\xi|>1\}} \mathcal{J}^{(i)}(d\xi, ds) \\
&\qquad + \int_0^t \int (\xi - \xi_i e_i) 1_{\{|\xi|\leq 1\}} \widetilde{\mathcal{J}}^{(i)}(d\xi, ds)\,.
\end{aligned}$$

where $\widetilde{\widetilde{\beta}}_i = \beta_i - e_i(\beta_i)_i$ and $\widetilde{\mathcal{J}}^{(i)}$ is the compensated jump measure.

Approximation of the jump part

- Introduce, for all $\underline{s} \in \mathbb{R}^m_{\geq 0}$,

$$\tilde{\mathcal{Z}}(\underline{s}) := \sum_{i=1}^{m} \tilde{Z}^{(i)}(s_i), \quad \tilde{\tilde{\mathcal{Z}}}(\underline{s}) := \sum_{i=1}^{m} \tilde{\tilde{Z}}^{(i)}(s_i) .$$

- Fix $M \in \mathbb{N}$ and consider the partition

$$\mathcal{T}_M := \left\{ \frac{k}{2^M}, \quad k \geq 0 \right\} .$$

- Define the following approximations on the partition $\mathcal{T}_M$:

$$\uparrow \tilde{\tilde{Z}}^{(i,M)}_t := \sum_{k=0}^{\infty} \tilde{\tilde{Z}}^{(i)}_{k/2^M} 1_{[\frac{k}{2^M}, \frac{k+1}{2^M})}(t) ,$$

$$\uparrow \tilde{\tilde{\mathcal{Z}}}^{(M)}(\underline{s}) := \sum_{i=1}^{m} \uparrow \tilde{\tilde{Z}}^{(i,M)}(s_i),$$

The proof

- Set

$$
\begin{aligned}
(t_0, \tau_0, x) &:= (0, 0, x),\\
\overleftarrow{\sigma} &:= (0, \dots, 0),\\
\overrightarrow{\sigma} &:= (\sigma_1^{(1,M)}, \dots, \sigma_1^{(i,M)}, \dots, \sigma_1^{(m,M)})
\end{aligned}
$$

- Solve

$$
\begin{cases}
\dot{\underline{\tau}}((0,0,x);t) = (x + \tilde{\tilde{\mathcal{Z}}})(\underline{\tau}((0,0,x);t)),\\
\underline{\tau}((0,0,x);t_0) = \tau_0\,,
\end{cases}
$$

for $t \in [0, t_1]$ where

$$
t_1 := \sup\{t > 0 \mid \underline{\tau}((t_0, \tau_0, x); t) \leq \overrightarrow{\sigma}\}\,.
$$

Remark There might be one or more indices i^*, where equality holds. Collect them in a set $I^* \subseteq \{1, \dots, m\}$.

- Update the values

$$\pi_{I^*}\overleftarrow{\sigma} := \pi_{I^*}\overrightarrow{\sigma},$$
$$\pi_{I^*}\overrightarrow{\sigma} := \pi_{I^*}\overrightarrow{\sigma}_{++},$$

where $\overrightarrow{\sigma}_{++}$ contains the next jumps of ${}^{\uparrow}\widetilde{\widetilde{Z}}^{(i,M)}$ for all $i \in I^*$ after $\overrightarrow{\sigma}_i$.

- Define

$$\tau_1 := \underline{\tau}((t_0, \tau_0, x); t_1)$$
$$x_1 := x + \Delta{}^{\uparrow}\widetilde{\widetilde{\mathcal{Z}}}^{(M)}(\overleftarrow{\sigma}).$$

- Solve

$$\begin{cases} \dot{\underline{\tau}}((t_1, \tau_1, x_1); t) = (x_1 + \widetilde{\widetilde{\mathcal{Z}}})(\underline{\tau}((t_1, \tau_1, x_1); t)), \\ \underline{\tau}((t_1, \tau_1, x_1); t_1) = \tau_1 , \end{cases}$$

for $t \in [t_1, t_2]$ where

$$t_2 := \sup\{t > t_1 \mid \underline{\tau}((t_1, \tau_1, x_1); t) \leq \overrightarrow{\sigma}\}.$$

- Define iteratively, for all $n \geq 1$

$$\begin{aligned} t_{n+1} &:= \sup\{t > 0 \mid \underline{\tau}((t_n, \tau_n, x_n); t) \leq \overrightarrow{\sigma}\}, \\ \tau_{n+1} &:= \underline{\tau}((t_n, \tau_n, x_n); t_{n+1}), \\ x_{n+1} &:= x_n + \Delta^{\uparrow} \tilde{\tilde{\mathcal{Z}}}^{(M)}(\overleftarrow{\sigma}), \end{aligned}$$

where, at each step $\overleftarrow{\sigma}$ and $\overrightarrow{\sigma}$ are updated.

Solution of the approximated problem

Theorem

There exists a solution of

$$\begin{cases} \dot{\underline{\tau}}^{(M)}((t_0,\tau_0,x);t) = (x + \tilde{\mathcal{Z}} + {}^{\uparrow}\tilde{\tilde{\mathcal{Z}}}^{(M)})(\underline{\tau}^{(M)}((t_0,\tau_0,x);t)), \\ \underline{\tau}^{(M)}((t_0,\tau_0,x);t_0) = 0\,. \end{cases}$$

Moreover it holds

$$\lim_{M\to\infty} \underline{\tau}^{(M)}((t_0,\tau_0,x);t) = \underline{\tau}((t_0,\tau_0,x);t)$$

where $\underline{\tau}$ solves

$$\begin{cases} \dot{\underline{\tau}}((t_0,\tau_0,x);t) = (x + \mathcal{Z})(\underline{\tau}((t_0,\tau_0,x);t)), \\ \underline{\tau}((t_0,\tau_0,x);t_0) = \tau_0\,. \end{cases}$$

Thank you for your attention

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