Regularity Results for Degenerate Kolmogorov Equation of Affine Type

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Set up

The state space: $D \subset \mathbb{R}^d$

The model: $X = (\Omega, (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{P}^x)_{x\in D_\Delta})$ *càdlàg*, time homogeneous, *conservative* Markov process in D

$$p_t(x, A) = \mathbb{P}^x(X_t \in A), \qquad t \geq 0,\ x \in D,\ A \in \mathcal{B}(D).$$

The problem: Regularity of

$$P_t f(x) := \mathbb{E}^x\Big[f(X_t)\Big],$$

$t \geq 0,\ x \in D,\ f \in \mathcal{M}$.

Option pricing

- $(X_t^x)_{t \in [0,T]}$
- f payoff

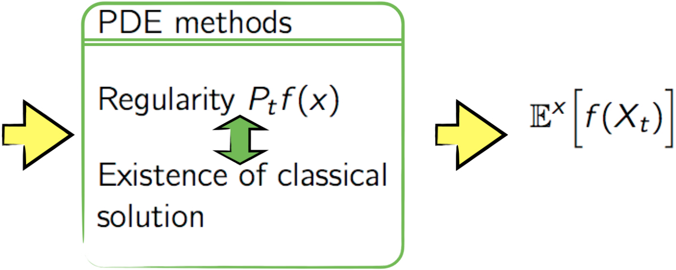
PDE methods

Probabilistic methods

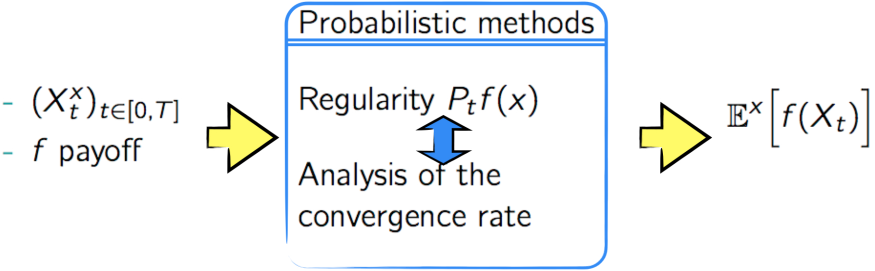
$\mathbb{E}^x \left[f(X_t) \right]$

Option pricing

- $(X_t^x)_{t \in [0,T]}$
- f payoff



Option pricing



The easier task: regularity in time

Suppose that, for all $f \in \mathcal{M}$ there exists a function $\mathcal{A}f$ such that

$$M_t^f := f(X_t) - f(x) - \int_0^t \mathcal{A}f(X_s)ds$$

is a true martingale.
Then, if $P_t|\mathcal{A}f| < \infty$ for all $t \in [0, T]$

$$P_t f(x) = f(x) + \int_0^t P_s \mathcal{A}f(x)ds.$$

Towards the Kolmogorov equation

$$P_t f(x) = f(x) + \int_0^t P_s \mathcal{A} f(x) ds \overset{?}{=} f(x) + \int_0^t \mathcal{A} P_s f(x) ds$$

Remark

Without additional regularity assumptions we cannot conclude that

$$\mathcal{A} P_t f = P_t \mathcal{A} f, \qquad t \geq 0, f \in \mathcal{M}.$$

We want...

... to deal with

- functions not necessarily bounded
- Markov processes with jumps...
- ....and possibly degenerate, e.g. with degeneracy in the diffusion coefficient as in stochastic volatility models

A paradigm: The CIR with jumps

$$
\begin{aligned}
X_t^x &= x + \int_0^t cX_s ds + \int_0^t \sqrt{X_s} dW_s \\
&\quad + \int_0^t \int_D h(\xi) \left(J^X(ds, d\xi) - \frac{d\xi}{\xi^2} X_s ds \right) \\
&\quad + \int_0^t \int_D (\xi - h(\xi)) J^X(ds, d\xi).
\end{aligned}
$$

We need...

... a methodology which does not requires ellipticity or non-degeneracy assumptions.

Example (A guiding example)

$$X_t^x := x + 2\int_0^t \sqrt{X_s}\, dW_s$$

$$L_x^t := 2t \sum_{k=1}^{N_{x/(2t)}} \mathbb{e}_k$$

$$X_t^x \overset{d}{=} L_x^t$$

The advantage

Differentiability in space for X

Differentiability in time for L

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Affine processes on the canonical state space

An *affine processes* on $D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ is a time homogeneous Markov process

$$X = (\Omega, (\mathcal{F}_t)_{t \geq 0}, (p_t)_{t \geq 0}, (X_t)_{t \geq 0}, (\mathbb{P}^x)_{x \in D_\Delta})$$

satisfying the following properties:

- (*stochastic continuity*) for every $t \geq 0$ and $x \in D$, $\lim_{s \to t} p_s(x, \cdot) = p_t(x, \cdot)$ weakly,
- (*affine property*) there exists $\Psi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}^d$ such that

$$\mathbb{E}^x\left[e^{\langle u, X_t \rangle}\right] = \int_D e^{\langle u, \xi \rangle} p_t(x, d\xi) = e^{\phi(t,u) + \langle \Psi(t,u), x \rangle}$$

for all $x \in D$ and $(t, u) \in \mathbb{R}_{\geq 0} \times \mathcal{U}$, with

$$\mathcal{U} = \left\{ u \in \mathbb{C}^d \mid e^{\langle u, x \rangle} \text{ is a bounded function on } D \right\}.$$

From affine processes to linear processes

Let $AP(D)$ be the space of affine processes with state space D. Define the map

$$\begin{aligned} \infty : AP(\mathbb{R}^{m}_{\geq 0} \times \mathbb{R}^{n}) &\rightarrow AP(\mathbb{R}^{m+1}_{\geq 0} \times \mathbb{R}^{n}) \\ X &\mapsto X^{\infty} \end{aligned}$$

Input: X with $\mathbb{E}^{x}\left[e^{\langle u, X_t \rangle}\right] = e^{\phi(t,u)+\langle x, \Psi(t,u)\rangle}$

Output: X^{∞} with $\mathbb{E}^{(1,x)}\left[e^{\left\langle u, X_t^{\infty}\right\rangle}\right] = e^{\langle (1,x), \Psi^{\infty}(t,u)\rangle}$

Linear structure

There exists a function $\Psi : \mathbb{R}_{\geq 0} \times \mathcal{U} \rightarrow \mathbb{C}^{d}$ such that

$$\mathbb{E}^{x}\left[e^{\langle u, X_t \rangle}\right] = \int_D e^{\langle u, \xi \rangle} p_t(x, d\xi) = e^{\langle \Psi(t,u), x \rangle}.$$

Generalized Riccati equations

On the set $\mathcal{Q} = \mathbb{R}_{\geq 0} \times \mathcal{U}$, the function Ψ satisfies the following system of generalized Riccati equations:

Generalized Riccati equations

$$\partial_t \Psi(t,u) = \mathbf{R}(\Psi(t,u)), \quad \Psi(0,u) = u$$

where for each $k = 1, \ldots, d$ the function $\mathbf{R}_k$ has the following Lévy-Khintchine form

$$\begin{aligned} \mathbf{R}_k(u) &= \langle \beta_k, u \rangle + \frac{1}{2} \langle u, \alpha_k u \rangle - \gamma_k \\ &\quad + \int_{D \setminus \{0\}} \left(e^{\langle u, \xi \rangle} - 1 - \left\langle \pi_{J \cup \{k\}} u, \pi_{J \cup \{k\}} h(\xi) \right\rangle \right) M_k(d\xi). \end{aligned}$$

Infinite divisibility in D

Let $\mathcal{C}$ the *convex cone* of continuous function $\eta : \mathcal{U} \to \mathbb{C}_+$ of type

$$\eta(u) = \langle \mathrm{b}, u \rangle + \frac{1}{2} \langle \pi_J u, \sigma \pi_J u \rangle + \int_{D \setminus \{0\}} \left(e^{\langle u, \xi \rangle} - 1 - \langle \pi_J u, \pi_J h(\xi) \rangle \right) \nu(d\xi). \qquad (*)$$

Definition

A distribution λ on D_Δ is *infinitely divisible* if and only if its Laplace transform takes the form $e^{\eta(u) - \mathrm{c}}$, where η has the form $(*)$ and $\mathrm{c} = \log \lambda(D)$.

$p_t(x, \cdot)$ is infinitely divisible in D!

Outline

A representation theorem

Let $(\mathcal{D}(\mathbb{R}_{\geq 0}; D_\Delta), (\mathcal{F}_t)_{t\geq 0}, (X_t)_{t\geq 0}, (\mathbb{P}^x)_{x\in D_\Delta})$ be a linear process on the canonical state space $D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$.

Proposition

For each fixed $t > 0$ and $x \in D$, there exists process $(L_s^{(t,x)})_{s\in[0,1]}$ such that:

1. $L_0^t = 0$,
2. for every $0 \leq s_1 \leq s_2 \leq 1$, the increment $L_{s_2}^{(t,x)} - L_{s_1}^{(t,x)}$ is independent of the family $(L_s^{(t,x)})_{s\in[0,s_1]}$ and it distributed as $X_t^{(s_2-s_1)x}$.

Moreover for any fixed $t \geq 0$, and $x \in D$ there exists a unique modification $\widetilde{L}^{(t,x)}$ of $L^{(t,x)}$ which is a Lévy process with càdlàg paths.

The proof

Apply Kolmogorov's existence Theorem with the convolution semigroup $(p_t(sx, \cdot))_{s \geq 0}$.

Chapman-Kolmogorov equations $\rightarrow$ Semigroup property in time

$$p_{s+t}(x, \cdot) = p_s \cdot p_t(x, \cdot) := \int p_t(x, dy) p_s(y, \cdot)$$

Linearity $\rightarrow$ Convolution property in space

$$p_t(x + y, \cdot) = p_t(x, \cdot) * p_t(y, \cdot)$$

Outline

The choice of the function space

$$C_F^\infty = \Big\{ f \in C^\infty(D), \text{ such that for all } \alpha \in \mathbb{N}^d \, \exists C_\alpha > 0, e_\alpha \in \mathbb{N} \mid \text{for all } x \in D \, |\partial_\alpha f(x)| \leq C_\alpha F_{e_\alpha}(x) \Big\}$$

Convergence rate for weak approximation

$$\hookrightarrow \; F_{e_\alpha}(x) = (1 + |x|^{e_\alpha})$$

Exponential semimartingale market models

$$\hookrightarrow \; F_{e_\alpha}(x) = \cosh(e_\alpha |x|)$$

The Result

Theorem

Let $f \in C_F^\infty$. Then the function $u : \mathbb{R}_{\geq 0} \times D \to \mathbb{R}$ defined by $u(t,x) = \mathbb{E}^x\Big[f(X_t)\Big]$ is smooth with all derivatives satisfying the following property

$$\text{for all } (t,x) \in [0,T] \times D, \quad |\partial^\alpha_{(t,x)} u(t,x)| \leq K_\alpha(T) F_{e_\alpha}(x),$$

where $K_\alpha(T)$ is a positive constant depending on the time horizon T and the order of derivative α and $e_\alpha > 0$.
Moreover it is a classical solution of the PIDE

$$\partial_t u(t,x) = \mathcal{A}u(t,x)$$
$$u(0,x) = f(x).$$

Dissect the theorem

part 1 $t \mapsto P_t f(x)$ is differentiable for all $x \in D$

$\hookrightarrow$ iterated Dynkin formula

part 2 $x \mapsto P_t f(x)$ is differentiable

$\hookrightarrow$ do a time-space shift

part 3 $(t, x) \mapsto P_t f(x)$ is differentiable with controlled growth.

Outline

Regularity in time: the martingale approach

Definition

Given a Markov process X the *extended generator* is an operator $\mathcal{A}$ with domain $\mathcal{D}(\mathcal{A})$ such that, for any $f \in \mathcal{D}(\mathcal{A})$ the process

$$M_t^f := f(X_t) - f(x) - \int_0^t \mathcal{A}f(X_s)ds$$

is a local martingale under $\mathbb{P}^x$ for every $x \in D$.

Integrability condition

Assumption A_F

For all $i = 1, \dots, d$

$$\gamma_i = 0 \text{ and } \int_{|\xi|>1} F_{e_\alpha}(\xi) M_i(d\xi) < \infty.$$

Lemma

Let X^x be an affine process satisfying A_F. Then, for all $f \in C_F^\infty$ the process

$$M_t^f := f(X_t) - f(x) - \int_0^t \mathcal{A} f(X_s) ds$$

is a true martingale.

Theorem

Let X^x be an affine process satisfying A_F. Then

(i) $\mathcal{A}C_F^\infty \subseteq C_F^\infty$,

(ii) *for any $f \in C_F^\infty$, $P_t f$ solves the Kolmogorov's equation*

$$\partial_t u(t,x) = \mathcal{A}u(t,x)$$
$$u(0,x) = f(x),$$

(iii) *for any $f \in C_F^\infty$ with $\nu \in \mathbb{N}$ the following expansion of the transition semigroup holds:*

$$\mathbb{E}^x\Big[f(X_t)\Big] = f(x) + \sum_{k=1}^{\nu} \frac{t^k}{k!}\mathcal{A}^k f(x) + \frac{t^{\nu+1}}{\nu!}\int_0^1 (1-s)^\nu \,\mathbb{E}^x\Big[\mathcal{A}^{\nu+1} f(X_{st})\Big]ds.$$

Sketch of the proof I

(i) From Itô formula one sees that, for all $f \in C_F^\infty$

$$|\mathcal{A}f(x)| \leq KF(x).$$

The bound also holds for all the derivatives.

(ii) From martingale property of M^f and (i) one gets (see [CKRT12])

- differentiability of $t \mapsto P_t f$
- commutative property $\mathcal{A}P_t = P_t\mathcal{A}$ for all $t \geq 0$
- Kolmogorov equation

Sketch of the proof II

(iii) by induction on ν :

$\nu = 0$

$$\mathbb{E}^x\Big[f(X_t)\Big] = f(x) + t\int_0^1 \mathbb{E}^x\Big[\mathcal{A}^1 f(X_{st})\Big]ds$$

$\nu = 1$

$$\begin{aligned}\mathbb{E}^x\Big[f(X_t)\Big] &= f(x) + t\mathcal{A}f(x) + \int_0^t (t-s)\,\mathbb{E}^x\Big[\mathcal{A}^2 f(X_s)\Big]ds \\ &= f(x) + t\mathcal{A}f(x) + \Big[\int_0^t P_s\mathcal{A}f(x)ds - t\mathcal{A}f(x)\Big] \\ &= f(x) + \int_0^t P_s\mathcal{A}f(x)ds\end{aligned}$$

$\nu = 2, \ldots$

Outline

Main idea

Prove regularity in space of the function

$$u(t,x) = \mathbb{E}^x\Big[f(X_t)\Big]$$

using regularity in time of the function

$$v^{(t,x)}(s,y) := \mathbb{E}\Big[f(L_s^{(t,x)} + y)\Big]$$

where $L^{(t,x)}$ is the Lévy process representing X_t^x.

A brief recap

Theorem

There exists a process $(L_s^{(t,x)})_{s\geq 0}$ taking values in D such that

1. *it has stationary and independent increments,*
2. *it is stochastically continuous,*
3. *it holds*

$$\mathbb{E}\left[e^{\left\langle u, L_s^{(t,x)}\right\rangle}\right] = e^{s\langle x, \Psi(t,u)\rangle} = \mathbb{E}^{sx}\left[e^{\langle u, X_t\rangle}\right].$$

Make things rigorous

step 1 Consider the decomposition

$$X^{x+hx_i} \overset{law}{=} X^x + \widetilde{X}^{hx_i},\ h > 0, i = 1, \ldots, d,$$

where $\widetilde{X}^{hx_i}$ is an independent copy of the process X starting from hx_i.

step 2 For fixed $(t, x) \in \mathbb{R}_{\geq 0} \times D$, $X_t^{x+hx_i}$ has the same distribution as the distribution of the Lévy process $L^{(t,x_i)}$ at time h starting from the initial random position $X_t^x \in D$.

step 3 From linearity in x,

$$u(t, x + hx_i) = \mathbb{E}^x\left[v^{(t,x_i)}(h, X_t)\right].$$

Iterated Dynkin formula

Proposition

Let $L^{(t,x)}$ be the Lévy process representing X^x.
For any fixed $t \geq 0$ and $x \in D$

$$Q_s^{(t,x)} f(y) = \mathbb{E}\Big[f(L_s^{(t,x)} + y)\Big]$$

$s \geq 0, y \in \mathbb{R}^d, f \in C_F^\infty$. Then

$$Q_s^{(t,x)} f = \sum_{k=0}^{\nu} \frac{s^k}{k!} (\mathcal{L}^{(t,x)})^k f + \mathcal{R}_\nu f,$$

with $\mathcal{R}_\nu$ is bounded uniformly in $s \in [0,1]$.

With the aid of the new representation

Theorem

Let X^x be an affine process satisfying the A_F. Then given a function $f \in C_F^\infty$, $P_t f$ is again in C_F^∞ for all $t \geq 0$.

Proof:

$$P_t f(x + h x_i) = \mathbb{E}^x \left[Q^{(t,x_i)} f(X_t) \right]$$

$$\partial_{x_i} u(t,x) = \mathbb{E}^x \left[\mathcal{L}^{(t,x_i)} f(X_t) \right]$$

$$\partial_x^\alpha u(t,x) = \mathbb{E}^x \left[(\mathcal{L}^{(t,x_1)})^{\alpha_1} \cdots (\mathcal{L}^{(t,x_d)})^{\alpha_d} f(X_t) \right].$$

Higher order derivatives

$$\alpha = (\underbrace{\alpha_0}_{\text{time derivatives}} \quad \overbrace{\overline{\alpha}}^{\text{space derivatives}})$$

By induction on $\alpha_0 := |\alpha| - |\overline{\alpha}|$

$\alpha_0 = 1$

$$\partial_t \partial_x^{\overline{\alpha}} u(t,x) = \mathcal{A} \partial_x^{\overline{\alpha}} u(t,x) \in C_F^\infty.$$

Suppose it holds for $\alpha_0 = 1, \dots, |\alpha| - |\overline{\alpha}| - 1$

$$\partial_t^{\alpha_0} \partial_x^{\overline{\alpha}} u(t,x) \quad = \quad \partial_t \left(\mathcal{A} \partial_t^{\alpha_0 - 1} \partial_x^{\overline{\alpha}} u(t,x) \right) \in C_F^\infty.$$

Thank you

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