# What is....an Affine Process?

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# Introduction


Figure: Google Inc.

# Black-Scholes Option Pricing

- $(S_t)_{t\geq 0}$ asset price,
- $(X_t)_{t\geq 0}$ log-price of the asset, i.e. $S_t = S_0 e^{X_t}$.

## Black-Scholes model

$$X_t = -\frac{1}{2}\sigma^2 t + \sigma B_t,$$

where

- $\sigma$ is the volatility parameter,
- $(B_t)_{t\geq 0}$ is a standard Brownian motion.

# Black Scholes formula

| Call Options | | | | | | | Expire at close Friday, May 20, 2011 |
|---|---|---|---|---|---|---|---|
| Strike | Symbol | Last | Chg | Bid | Ask | Vol | Open Int |
| 400.00 | GOOG110521C00400000 | 135.50 | 0.00 | 141.00 | 144.20 | 8 | 22 |
| 420.00 | GOOG110521C00420000 | 104.53 | 0.00 | 121.20 | 124.60 | 1 | 33 |
| 430.00 | GOOG110521C00430000 | 95.80 | 0.00 | 111.20 | 114.70 | 2 | 2 |
| 440.00 | GOOG110521C00440000 | 85.70 | 0.00 | 101.20 | 104.30 | 4 | 16 |

Figure: Call Options on Google Inc.

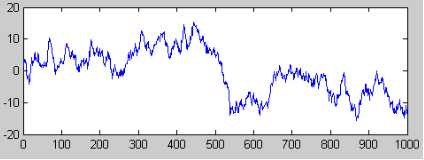
- A European call option on an asset $S_t$, paying no dividends, with maturity date T and strike price K is defined as a contingent claim with payoff $\max(S_T - K, 0)$ at maturity,
- The Black-Scholes formula for the value of this call option is:

$$C_{BS}(S_t, K, \tau, \sigma) = S_t N(d_1) - KN(d_2),$$
$$d_1 = \frac{-\log x + \tau \frac{\sigma^2}{2}}{\sigma\sqrt{\tau}}, \quad d_2 = \frac{-\log x - \tau \frac{\sigma^2}{2}}{\sigma\sqrt{\tau}},$$

where $\tau = T - t, x = K/S_t$ and $N(u)$ is the normal cumulative distribution function.

# Black Scholes limitations (1)

Figure: Evolution of SLM (NYSE), 5d, 1m and 1y




- Brownian motion paths are continuous,
- Scale invariance property for Brownian motion: if $(B_t)_{t\geq 0}$ is a Brownian motion, then the process $X_t = \frac{1}{a} B_{a^2 t}$, $a > 0$ is also a Brownian motion.

# Black Scholes limitations (2)

Gaussian patterns are in contradiction with market reality:

- Distributions are characterized by heavy tails and high peaks,
- If $C_t^*(T, K)$ are the market prices, there exists a unique volatility parameter $\sigma_{BS}(T, K)$ such that the corresponding Black-Scholes prices match the market price:

$$\exists ! \sigma_{BS}(K, T) > 0 \; : \; C_{BS}(S_t, K, \tau, \sigma_{BS}(T, K)) = C_t^*(T, K).$$

**Implied volatility**

$\sigma_{BS} \; : \; (K, T) \to \sigma_{BS}(K, T).$

- Black-Scholes model predicts a flat profile:

$$\sigma_{BS}(K, T) = \sigma.$$

# Some market evidences

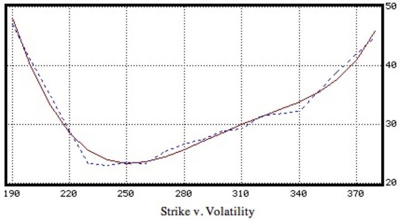



Figure: Corn Option Analysis: Volatility Skew

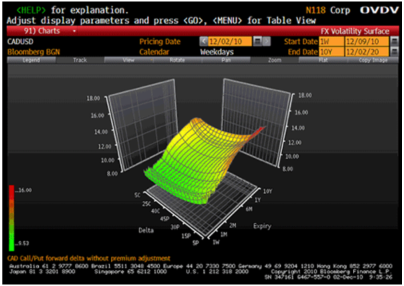



Figure: Implied Volatility Surface of CAD/USD Cross

# Add more features.....

- Remove continuity assumptions on sample paths $\rightsquigarrow$ jump diffusion models,
- Add more source of uncertainty $\rightsquigarrow$ stochastic volatility models,
- Dependence structure between different assets $\rightsquigarrow$ covariance processes, i.e. matrix-valued processes.

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# Canonical Affine Processes

# Add more features.....jumps

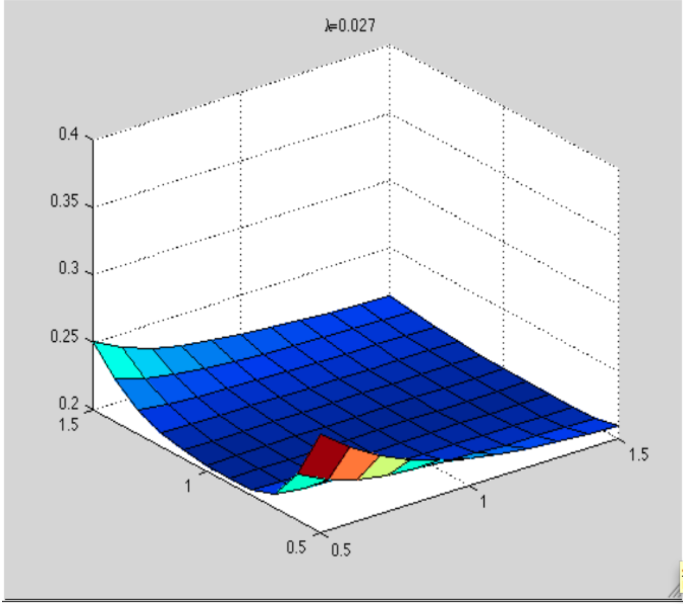
## Merton jump diffusion model

$$X_t = \left(-\frac{1}{2}\sigma^2 - \lambda k\right) t + \sigma B_t + \sum_{j=1}^{N_t} Y_k,$$

where

- $\sigma$ is the volatility parameter,
- $(B_t)_{t\geq 0}$ is a standard Brownian motion process,
- $(N_t)_{t\geq 0}$ is a Poisson process with intensity $\lambda$,
- $(Y_j)_{j\geq 0}$ are i.i.d. random variables with distribution $\mathcal{N}(\mu, \delta^2)$,
- $k = e^{\mu+\frac{1}{2}\delta^2} - 1$.

# Merton implied volatility surface



# Add more features....random clock

## Variance Gamma model

$$X_t = \frac{t}{\nu}\log\left(1 - \theta\nu - \frac{\sigma^2\nu}{2}\right) + \theta\Gamma_t + \sigma B_{\Gamma_t},$$

where

- $(B_t)_{t\geq 0}$ is a Brownian motion,
- $(\Gamma_t)_{t\geq 0}$ is an independent gamma process with unit mean and variance $\nu$.

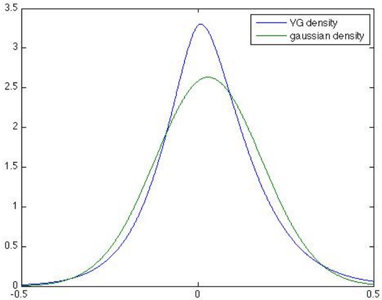



Figure: Comparison between VG density and gaussian density with same mean and variance.

# Add more features..... stochastic volatility

## The Heston Model

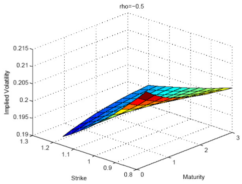
$$
\begin{aligned}
dX_t &= -\frac{V_t}{2}dt + \sqrt{V_t}dB_t, \\
dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t}d\widetilde{B}_t, \\
dB_t d\widetilde{B}_t &= \rho dt,
\end{aligned}
$$

where

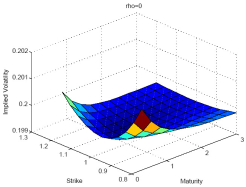
- $(S_t)_{t\geq 0}$ and $(V_t)_{t\geq 0}$ are price and volatility processes,
- $(B_t)_{t\geq 0}$ and $(\widetilde{B}_t)_{t\geq 0}$ are Brownian motions with correlation $\rho$,
- $\theta$ is long-run mean, $\kappa$ is the rate of reversion and $\sigma$ is volatility of volatility.

# Impact of $\rho$ on volatility surfaces

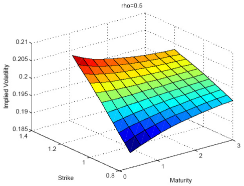
$\rho = -0.5$




$\rho = 0$




$\rho = 0.5$



# Definition

A stochastic process $X$ is called a *canonical affine process*, if it is

- a time-homogeneneous Markov process,
- stochastically continuous,
- takes values in $D = \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$,
- has the following property:

## Affine Property

There exists functions $\phi$ and $\psi$, taking values in $\mathbb{C}$ and $\mathbb{C}^{m+n}$ respectively, such that

$$\mathbb{E}_x \left[ e^{\langle X_t, u \rangle} \right] = e^{\phi(t,u) + \langle x, \psi(t,u) \rangle},$$

for all $x \in D$, and for all $(t, u) \in \mathbb{R}_{\geq 0} \times \mathcal{U}$, where

$$\mathcal{U} = \{u \in \mathbb{C} \mid \mathcal{R}e\langle x, u \rangle \leq 0 \text{ for all } x \in D\}.$$

# Regularity

An Affine Process is called regular, if the derivatives

$$F(u) := \partial_t \phi(t,u)\big|_{t=0}, \quad R(u) := \partial_t \psi(t,u)\big|_{t=0}$$

exist, and are continuous at $u = 0$.

**Theorem (Keller-Ressel, Teichmann, Schachermayer (2009))**

Every canonical Affine Process is regular.

**Theorem (Duffie et al. (2003))**

If $(X_t)_{t\geq 0}$ is a regular affine process, then $\phi$ and $\psi$ satisfy the generalized Riccati equations

$$\begin{aligned} \partial_t \phi(t,u) &= F(\psi(t,u)), \quad \phi(0,u) = 0, \\ \partial_t \psi(t,u) &= R(\psi(t,u)), \quad \psi(0,u) = u. \end{aligned}$$

# Examples (1)

- The functions $F : \mathcal{U} \to \mathbb{C}$ and $R : \mathcal{U} \to \mathbb{C}^d$ are explicitly described in terms of model parameters,
- $F(u)$ represents the state-independent dynamic, and $R(u)$ the state-dependent dynamic of the process.

**Brownian Motion**

$dX_t = bdt + \sigma dB_t,$

$$F(u) = bu + \frac{1}{2}\sigma^2 u^2$$
$$R(u) = 0$$

**Vasicek**

$dX_t = (b + \beta X_t)dt + \sigma dB_t,$

$$F(u) = bu + \frac{1}{2}\sigma^2 u^2$$
$$R(u) = \beta u$$

**CIR**

$dX_t = (b + \beta X_t)dt + \sigma \sqrt{X_t} dB_t,$

$$F(u) = bu$$
$$R(u) = \beta u + \frac{1}{2}\sigma^2 u^2$$

# Example (2)

## Heston

$$
\begin{aligned}
dV_t &= (b + \beta V_t)dt + \sigma\sqrt{V_t}dB_t^1,\\
dX_t &= -\tfrac{1}{2}V_t dt + \sqrt{V_t}(\rho dB_t^1 + \sqrt{1-\rho^2}dB_t^2),\\
F(u_1,u_2) &= (b,0)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\\
R_1(u_1,u_2) &= \tfrac{1}{2}(u_1,u_2)\begin{pmatrix} \sigma^2 & \rho\sigma \\ \rho\sigma & 1 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + (\beta, -\tfrac{1}{2})\begin{pmatrix} u_1 \\ u_2 \end{pmatrix},\\
R_2(u_1,u_2) &= 0.
\end{aligned}
$$

# Example (3)

- Continuous-state branching processes exhibit the aforementioned ‘affine property’.

**Example:** $dY_t = \sqrt{2Y_t}dB_t \qquad Y_0 = x \geq 0$

$$\mathbb{E}\left[e^{uY_t}\right] = e^{x\frac{u}{1-ut}}, \quad u \in \mathbb{C} \text{ with } \mathcal{R}e(u) < \frac{1}{t}.$$

- $\phi(t, u) = 0$,
- $\psi(t, u)$ is a Möbius transformation.

**Remark**

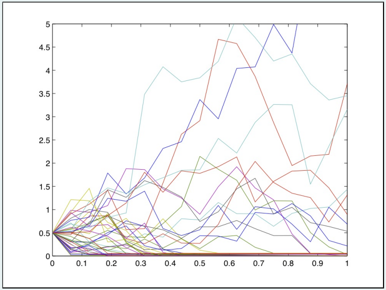
The class of CBI-processes is precisely the class of affine processes with state space $\mathbb{R}^d_{\geq 0}$.

# Path Approximation and Option Pricing

# Path Approximation

## A Motivating Example

$$dY_t = \sqrt{2Y_t}dB_t,$$
$$Y_0 = x \geq 0.$$




- The diffusion term, being decreasing to zero as $Y_t$ approaches the origin, prevents $(Y)_{t\geq 0}$ from taking negative values. This feature can be attractive in interest rate modeling,
- The main difficulty in discretization is located at the boundary of the state space cones, where the **vector fields lack the Lipschitz property**.

# European Options

- A fundamental computational operation in any asset pricing model is the pricing of European options.
- Pricing of such an option amounts to calculating the expectation

$$C(T, K) = \mathbb{E}[max(S_T - K, 0)],$$

under the risk-neutral measure.

## Numerical methods in Option pricing

1. if the distribution of $S_t$ is analytically known $\rightsquigarrow$ Numerical quadrature,
2. if the Laplace-Fourier transform is analytically know $\rightsquigarrow$ Fourier Methods,
3. Monte Carlo Simulation.

# Example: Option pricing in Merton Jump Diffusion Model

## Density Function

$$f_M(x) = \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} \mathcal{N}(x),$$

where $\mathcal{N}(x) = \frac{1}{\sqrt{2\pi(\sigma^2 t + n\delta^2)}} \exp\left(-\frac{\left(x+(\frac{1}{2}\sigma^2+\lambda k)t-nm\right)^2}{2(\sigma^2 t+n\delta^2)}\right).$

## Characteristic function

$$\mathbb{E}[e^{iuX_t}] = \exp\left\{t\left(iu\gamma - \frac{1}{2}\sigma^2 u^2 + \lambda(e^{ium-\frac{1}{2}\delta^2 u^2} - 1)\right)\right\}.$$

## Option pricing

- $C = \sum_{n\geq 0} \frac{e^{-\lambda\tau}(\lambda\tau)^n}{n!} C_{BS}(t, S_n, \sigma_n)$
  with $\sigma_n^2 = \sigma^2 + \frac{n\delta^2}{\tau}$
  $S_n = S\exp\left(nm + \frac{n\delta^2}{2} - \lambda\tau e^{m+\delta/2} + \lambda\tau\right),$
- $C = S - \frac{\sqrt{SK}}{\pi} \int_0^\infty \mathcal{R}e\left(e^{iuk}\mathbb{E}\left[e^{iu(u-\frac{i}{2})}\right]\right)\frac{du}{u^2+\frac{1}{4}}.$

# Multivariate Affine Stochastic Volatility Model

# Affine processes on positive semidefinite matrices

- $\mathcal{S}_d$ : symmetric $d \times d$-matrices equipped with scalar product $\langle x, y\rangle = Tr(xy)$,
- $\mathcal{S}_d^+$ : cone of symmetric $d \times d$-positive semidefinite matrices,
- $\mathcal{S}_d^{++}$ : interior of $S_d^+$ in $S_d$.

## Example: Wishart Process

$$dX_t = (b + MX_t + X_t M^T)dt + \sqrt{X_t} dB_t \Sigma + \Sigma^T dB_t^T \sqrt{X_t}, \quad X_0 = x$$

- $x \in S_d^+$,
- $M$ $d \times d$ invertible matrix,
- $\Sigma$ $d \times d$ invertible matrix,
- $b \in \mathcal{S}_d^+$ such that $b - (d-1)\Sigma^T\Sigma \in \mathcal{S}_d^+$,
- $(B_t)_{t \geq 0}$ is a $d \times d$ matrix of Brownian motion.

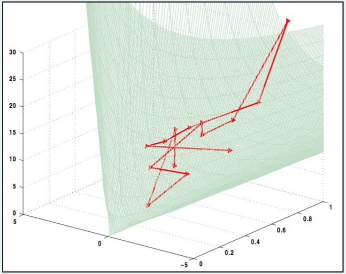



Figure: Wishart Process path

# Definition

A stochastic process $(X_t, Y_t)_{t\geq 0}$ is called a *Multivariate Affine Stochastic Volatility Model*, if it is

- a time-homogeneneous Markov process,
- stochastically continuous,
- takes values in $D = \mathcal{S}_d^+ \times \mathbb{R}^d$,
- has the following property:

## Affine Property

There exists functions $\Phi$ and $\psi$, such that

$$\mathbb{E}_{x,y}\left[e^{Tr(uX_t)+v^T Y_t}\right] = \Phi(t,u,v)e^{Tr(\psi(t,u,v)x)+v^T y},$$

for all $(x, y) \in D$, and for all $(t, u, v) \in \mathcal{Q}$, where

$$\mathcal{U} = \left\{(t,u,v) \in \mathbb{R}_{\geq 0} \times \mathcal{S}_d + i\mathcal{S}_d \times \mathbb{C}^d \mid \mathbb{E}_{x,y}\left[e^{\left|Tr(uX_t)+v^T Y_t\right|}\right] < \infty\right\}.$$

**Theorem (Cuchiero (2011))**

Let $(\tau, u, v) \in \mathcal{Q}$ such that $\mathbb{E}_{0,0}\left[e^{Tr(uX_\tau)+v^T Y_\tau}\right] \neq 0$.

- The derivatives

$$F(u,v) := \partial_t \Phi(t,u,v)\big|_{t=0} \text{ and } R(u,v) := \partial_t \psi(t,u,v)\big|_{t=0}$$

  exist are continuous in $(u, v)$.

- For $t \in [0, \tau)$, $\Phi$ and $\psi$ satisfy

$$\begin{aligned} \partial_t \Phi(t,u,v) &= \Phi(t,u,v) F(\psi(t,u,v), v), & \Phi(0,u,v) &= 1, \\ \partial_t \psi(t,u,v) &= R(\psi(t,u,v), v), & \psi(0,u,v) &= u. \end{aligned}$$

# Example: Multivariate Heston Model

$$dX_t = (b + MX_t + X_t M^T)dt + \sqrt{X_t} dB_t \Sigma + \Sigma^T dB_t^T \sqrt{X_t},$$
$$dY_t = -\frac{1}{2} X_t^{diag} dt + \sqrt{X_t} d\widetilde{B}_t,$$

where

- $X_t^{diag}$ is the vector containing the diagonal entries of $X_t$,
- $(\widetilde{B}_t)_{t\geq}$ is a $\mathbb{R}^d$-valued Brownian motion correlated with $(B_t)_{t\geq}$ with correlation $\rho \in \mathbb{R}^d$.

$$\begin{aligned} F(u,v) &= Tr(bu), \\ R(u,v) &= 2u\Sigma^T\Sigma u + \frac{1}{2}v^T v + u(\Sigma^T \rho v^T + M) + (v\rho^T\Sigma + M^T)u \\ &\quad -\frac{1}{2}diag(v), \end{aligned}$$

where $diag(v)$ is the $d \times d$ matrix with $v$ on the main diagonal.

Thank you for your attention